Math 561: Foundations of Combinatorics
Lecturer: Prof. Sara Billey

Problem Set #2

due Wednesday, October 9, 2019

Reading: In EC 1, Chapter 1, read Sections 1.2–1.5 As time permits, also try reading Sagan Chapter 1, Sections 1.2-=1.5 and Ardila Chapter 2 Sections 2.1–2.3. We won't cover Ardila's Section 2.4 until the end of the quarter (and only as time permits)!

Recommended Problems: Give each of these problems careful consideration before reading the solutions: EC1, Chapter 1: 3(f and g), 10, 13, 15(c), 19, 23, 24, 57, 58, 60.

Homework Problems: For each of the problems below, explain your answer fully. No credit will be given for a simple statement of the answer. Each problem is worth 10 points unless otherwise specified.

1. If $f(x)$ is the ordinary generating function of the sequence $\{a_n\}_{n\geq 0}$ then express simply, in terms of $f(x)$, the ordinary generating functions for the following sequences:
   (a) $\{\alpha a_n + c\}_{n\geq 0}$, α, c constants.
   (b) $a_0, 0, a_2, 0, a_4, 0, a_6, 0, a_8, 0, \ldots$
   (c) $\{a_{n+2} + 3a_{n+1} + a_n\}_{n\geq 0}$
   (d) $0, 0, 1, a_3, a_4, a_5, \ldots$

2. Let $f(x) = \sum_{i=0}^{k} a_i x^i$ be a polynomial with nonnegative integer coefficients. Define a random variable X_f which takes on values $\{0, 1, 2, \ldots, k\}$ with probability
$$P(X_f = i) = \frac{a_i}{f(1)}.$$
What is the expected values of X_f as a function of the a_i's. What is the variance of X_f as a function of the a_i's?

3. Exercise 12 from Chapter 1 of EC1.

4. Exercise 14 from Chapter 1 of EC1.

5. Exercise 17 from Chapter 1 of EC1.

6. Exercise 29 from Chapter 1 of EC1.

7. Exercise 58(a) from Chapter 1 of EC1.

8. Use RSK to give a combinatorial proof that 321-avoiding permutations are Catalan objects.

9. Compute the number of ways to write the permutations 21, 321, 4321 as a minimal product of adjacent transpositions (reduced expressions). Extend this sequence as far as possible and use Sloane's On-line Encyclopedia of Integer sequences to conjecture an explicit formula for the number reduced expressions of the permutation $n, n-1, \ldots, 1$. (Bonus: Prove your conjecture.)