Math 561: Foundations of Combinatorics
Lecturer: Prof. Sara Billey

Problem Set #1

due Wednesday, October 2, 2019

Reading: Let's compare books this week! In EC 1, Chapter 1, read Sections 1.1 and 1.2 specifically omitting pages 10-14 for now. Also read the Appendix on Graph Theory Terminology pages 571–573. As time permits, also try reading Sagan Chapter 1, Sections 1.1 and 1.2 and Ardila Chapter 1, plus Chapter 2, Sections 2.1 and 2.2.

Recommended Problems: Give each of these problems careful consideration before reading the solutions: In EC1, Chapter 1: 2, 3ab, 8b, 9a.

Homework Problems: For each of the problems below, explain your answer fully. No credit will be given for a simple statement of the answer. Each problem is worth 10 points unless otherwise specified.

1. Use the Pigeonhole Principle to prove that the decimal expansion of any rational number m/n eventually is repeating.

2. (a) Prove combinatorially $\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3} = n^3$.

   (b) Prove combinatorially $\binom{0}{k} + \binom{1}{k} + \ldots + \binom{n}{k} = \sum_{j=0}^{n} \binom{j}{k} = \binom{n+1}{k+1}$ where n, k are any fixed non-negative integers.

   (c) Find a simple form for $1^3 + 2^3 + \cdots + n^3$.

3. Find the generating function for each of the following sequences and restate the problems in combinatorial language.

   (a) The number of collections of n balloons with a multiple of three red ones, at least 2 blue ones and no more than 5 green ones. Assume balloons of the same color are indistinguishable.

   (b) The number of ways one can give n children each one balloon such that the entire set of balloons has a multiple of three red ones, at least 2 blue ones and no more than 5 green ones.

4. Prove that for any $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$ there exist two of the integers a_i and a_j with $i \neq j$ such that $a_i - a_j$ is divisible by n.

5. How many ways can coins be stacked in the plane such that the bottom row consists of n consecutive coins? For example, there are five ways to do this if $n = 3$:

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                                  0
         0         0     0 0     0 0
0 0 0   0 0 0   0 0 0   0 0 0   0 0 0
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6. Find an explicit formula for the number of cards in a card house of depth 1 and with n levels. For example, it takes 2 cards to make a small house of one level built by leaning them against each other. To build a second level, build two small houses together and put a top on it across both peaks. Then build a second small house on top.

7. (30 pts) Prove any 3 parts to Problem 35 in Chapter 1 of EC1 rated 2 or 2+. Try each part before consulting the solutions. Additional parts count as bonus problems for additional credit.

8. Let a_n be the number of ways to write the positive integer n as the sum of powers of 2, disregarding order.

   (a) Find a_n for $n = 1, 2, 3, \dots, 10$

   (b) Give a generating function for a_n.

   (c) Use your answers in parts a) and b) to conjecture and prove a result comparing a_{2n} and a_{2n+1}.

9. (Bonus (20 pts)): Let b_n be the number of different rooted binary trees with n unlabeled leaves. Show

$$b_n = \sum_{\lambda} \frac{\prod_{i=2}^{\ell(\lambda)} (2(\lambda_i + \cdots + \lambda_{\ell(\lambda)}) - 1)}{z_\lambda},$$

where the sum is over all decreasing sequences $\lambda = (\lambda_1, \dots, \lambda_k)$ which sum to n and each part λ_i is a power of 2. The notation $\ell(\lambda) = k$ is the number of nonzero parts of λ, and

$$z_\lambda = (1^{m_0} 2^{m_1} 4^{m_2} 8^{m_3} \cdots)(m_0! m_1! m_2! \cdots)$$

if λ has m_i parts equal to 2^i for all integers $i \leq 0$.