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Pattern avoidance and the Bruhat order

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Abstract

The structure of order ideals in the Bruhat order for the symmetric group is elucidated via permutation patterns. The permutations with boolean principal order ideals are characterized. These form an order ideal which is a simplicial poset, and its rank generating function is computed. Moreover, the permutations whose principal order ideals have a form related to boolean posets are also completely described. It is determined when the set of permutations avoiding a particular set of patterns is an order ideal, and the rank generating functions of these ideals are computed. Finally, the Bruhat order in types B and D is studied, and the elements with boolean principal order ideals are characterized and enumerated by length.

Keywords: Bruhat order; Coxeter group; Pattern; Reduced decomposition; Order ideal; Interval; Boolean poset

1. Introduction

This paper studies the interplay between the Bruhat order and permutation patterns, with particular emphasis on these relationships in the symmetric group. The principal order ideals in particular are considered, and several results are described which emphasize the relationship between permutation patterns and reduced decompositions. The final section of the paper discusses the Bruhat order for types B and D, although in less depth than the type A discussion.

The finite Coxeter groups of types A, B, and D have combinatorial interpretations as permutations, signed permutations, and signed permutations with certain restrictions. The combinatorial aspects of Coxeter groups are treated in [1]. Although these groups are classical objects with a bountiful literature, there are still many open questions, particularly in reference to patterns and the Bruhat order.

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Following the work of Simion and Schmidt in [12], there has been a surge of interest in permutation patterns. Although many intriguing results have been shown, some of the most basic questions, such as how many permutations avoid a given pattern, remain unanswered. However, recent work (see [18]) has uncovered connections between reduced decompositions and permutation patterns that may prove useful to resolving some of these issues.

The Bruhat order is a partial ordering of Coxeter group elements, and it plays a remarkably significant role in the study of these groups. Somewhat surprisingly, very little is known about its structure, particularly in terms of its order ideals and intervals. The results presented here elucidate some pattern-related facts about this structure. When combined with the relationship between reduced decompositions and patterns in [18], these are significant steps towards understanding the more general structural aspects of this partial order. This paper primarily examines the structure of the Bruhat order of the symmetric group.

Section 4 classifies all permutations with boolean principal order ideals. As shown in Theorem 4.3, these are exactly those permutations that avoid the patterns 321 and 3412. The permutations with this property are enumerated by length in Corollary 4.5. Additionally, permutations with "nearly boolean" principal order ideals are discussed, along with the size and description of their ideals.

A more general classification occurs in Section 5. There the permutations with principal order ideals isomorphic to a power of $B(w_0^{(k)})$, for $k \geqslant 3$, are entirely classified as those in which every inversion is in exactly one decreasing subsequence of length k. This characterization (Theorem 5.6) is again stated in terms of patterns.

Section 6 examines sets of permutations avoiding either one or two patterns, and determines exactly when these sets are order ideals in the Bruhat order. This property holds in only a few situations, each of which can be enumerated by length.

Expanding on the results of Section 4, Section 7 examines the Bruhat order for the finite Coxeter groups of types B and D. In particular, those elements with boolean principal order ideals are characterized in Theorems 7.4 and 7.7. Once again, permutation patterns emerge, although now for signed permutations, and the avoidance of certain patterns is equivalent to having a boolean principal order ideal. While the case for type A required avoiding only two patterns, it is necessary to avoid ten patterns in type B, and twenty patterns must be avoided to have a boolean principal order ideal in type D. For types B and D, the elements avoiding these patterns are enumerated by length in Corollaries 7.6 and 7.9.

2. Definitions and background

Let $\mathfrak{S}_n$ be the group of permutations on n elements, and let $[n]$ denote the set of integers $\{1, \ldots, n\}$. An element $w \in \mathfrak{S}_n$ is the bijection on $[n]$ mapping $i \mapsto w(i)$. A permutation will be written in one-line notation as $w = w(1)w(2)\cdots w(n)$.

Example 2.1. $4213 \in \mathfrak{S}_4$ maps $1 \mapsto 4$, $2 \mapsto 2$, $3 \mapsto 1$, and $4 \mapsto 3$.

Definition 2.2. An *inversion* in w is a pair (i, j) such that $i < j$ and $w(i) > w(j)$.

Let $[\pm n]$ denote the set of integers $\{\pm 1, \ldots, \pm n\}$. For ease of notation, a negative sign may be written beneath an integer: $\underline{i} := -i$.

Definition 2.3. A *signed permutation* of $[\pm n]$ is a bijection w with the requirement that $w(\underline{i}) = \underline{w(i)}$. Let $\mathfrak{S}_n^B$ denote the signed permutations of $[\pm n]$.

An element $w \in \mathfrak{S}_n^B$ is entirely defined by $w(1), \ldots, w(n)$. Therefore one-line notation will again be used, although some values may now be negative.

Example 2.4. $\underline{4}21\underline{3} \in \mathfrak{S}_4^B$ maps $\pm 1 \mapsto \mp 4$, $\pm 2 \mapsto \pm 2$, $\pm 3 \mapsto \pm 1$, and $\pm 4 \mapsto \mp 3$.

The Coxeter groups studied here are the finite Coxeter groups of types A, B, and D. More thorough discussions of general Coxeter groups appear in [1] and [10].

Define involutions, called *simple reflections*, on $[\pm n]$ as follows:

$$s_i := 1 \cdots (i-1)(i+1)i(i+2) \cdots n \quad \text{for } i \in [n-1];$$
$$s_0 := \underline{1}2 \cdots n; \quad \text{and}$$
$$s_{1'} := s_0 s_1 s_0 = \underline{2}\underline{1}3 \cdots n.$$

These definitions indicate that the following braid relations hold:

$$s_i s_j = s_j s_i \quad \text{for } i, j \in [0, n-1] \text{ and } |i-j| > 1; \tag{1}$$
$$s_{1'} s_i = s_i s_{1'} \quad \text{for } i \in \big([n-1] \setminus \{2\}\big); \tag{2}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for } i \in [n-2]; \tag{3}$$
$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0; \quad \text{and} \tag{4}$$
$$s_{1'} s_2 s_{1'} = s_2 s_{1'} s_2. \tag{5}$$

Note that Eq. (4) follows from Eq. (2).

The finite Coxeter group of type A is the symmetric group $\mathfrak{S}_n$, for some n. This group is generated by $\{s_1, \ldots, s_{n-1}\}$. The simple reflections $\{s_0, s_1, \ldots, s_{n-1}\}$ generate the finite Coxeter group of type B, which is the hyperoctahedral group $\mathfrak{S}_n^B$, for some n. The finite Coxeter group of type D is the subgroup $\mathfrak{S}_n^D$ of $\mathfrak{S}_n^B$ consisting of signed permutations whose one-line notation contains an even number of negative values. This group is generated by $\{s_{1'}, s_1, \ldots, s_{n-1}\}$.

Definition 2.5. Let W be a Coxeter group generated by the simple reflections $\mathcal{S}$. For $w \in W$, if $w = s_{i_1} \cdots s_{i_\ell}$ and ℓ is minimal among all such expressions, then the string of indices $i_1 \cdots i_\ell$ is a *reduced decomposition* of w and ℓ is the *length* of w, denoted $\ell(w)$. The set $R(w)$ consists of all reduced decompositions of w.

The permutation $w_0 := n \cdots 21 \in \mathfrak{S}_n$ is the longest element in $\mathfrak{S}_n$, and $\ell(w_0) = \binom{n}{2}$. If n is unclear from the context, this may be denoted $w_0^{(n)}$. Analogously, the longest element in $\mathfrak{S}_n^B$ is $\underline{1}\underline{2} \cdots \underline{n}$, and the longest element in $\mathfrak{S}_n^D$ is $\underline{1}\underline{2} \cdots \underline{n}$ if n is even, and $1\underline{2}\underline{3} \cdots \underline{n}$ if n is odd.

Definition 2.6. A consecutive substring of a reduced decomposition is a *factor*.

Multiplication here follows the standard that a function is to the left of its input. Thus, if $i \in [n-1]$, then $s_i w$ interchanges the positions of the values i and $i+1$ (and $\underline{i}$ and $\underline{i+1}$) in w, whereas $w s_i = w(1) \cdots w(i+1) w(i) \cdots w(n)$.

The classical notion of (unsigned) permutation pattern avoidance is as follows.

Definition 2.7. Fix $w \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$ for $k \leqslant n$. The permutation w *contains the pattern* p, or *contains a* p*-pattern*, if there exist $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that $w(i_1) \cdots w(i_k)$ is in the same relative order as $p(1) \cdots p(k)$. That is, $w(i_h) < w(i_j)$ if and only if $p(h) < p(j)$. If w does not contain p, then w *avoids* p, or is p*-avoiding*.

In [18], reduced decompositions are analyzed in conjunction with pattern containment in $\mathfrak{S}_n$. This coordinated approach yields a number of significant results, and the main theorem of [18] is the vexillary characterization below. Vexillary permutations have several equivalent definitions. The original definition, formulated by Lascoux and Schützenberger, is in terms of pattern avoidance.

Definition 2.8. A permutation is *vexillary* if it is 2143-avoiding.

In order to state the vexillary characterization of [18], the following definition is necessary.

Definition 2.9. The *shift* of a string $\boldsymbol{i} = i_1 \cdots i_\ell$ by $M \in \mathbb{N}$ is

$$\boldsymbol{i}^M := (i_1 + M) \cdots (i_\ell + M).$$

Theorem *(Vexillary characterization). The permutation p is vexillary if and only if, for every permutation w containing a p-pattern, there exists a reduced decomposition $\boldsymbol{j} \in R(w)$ containing a shift of some $\boldsymbol{i} \in R(p)$ as a factor.*

The definition of patterns in signed permutations requires an extra clause.

Definition 2.10. Fix $w \in \mathfrak{S}_n^B$ and $p \in \mathfrak{S}_k^B$ for $k \leqslant n$. The permutation w *contains the pattern* p if there exist $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that

(1) $w(i_j)$ and $p(j)$ have the same sign; and
(2) $|w(i_1)| \cdots |w(i_k)|$ is in the same relative order as $|p(1)| \cdots |p(k)|$.

If w does not contain p, then w *avoids* p, or is p*-avoiding*.

Example 2.11. Let $w = \underline{4}21\underline{3}$, $p = \underline{3}1\underline{2}$, $q = 31\underline{2}$, and $r = 132$. Then $\underline{4}1\underline{3}$ and $\underline{4}2\underline{3}$ are both occurrences of p in w. The signed permutation w is q- and r-avoiding.

If w has a p-pattern, with $\{i_1, \ldots, i_k\}$ as in Definitions 2.7 and 2.10, then $w(i_1) \cdots w(i_k)$ is an *occurrence* of p in w. Define $\langle p(j)p(j+1) \cdots p(j+m) \rangle$ to be $w(i_j)w(i_{j+1}) \cdots w(i_{j+m})$. Different occurrences of p will be distinguished by subscripts, as in $\langle\ \rangle_1$ and $\langle\ \rangle_2$.

Example 2.12. Let $w = 7413625$, $p = 1243$, and $q = 1234$. Then 1365 is an occurrence of p, with $\langle 1 \rangle = 1$, $\langle 24 \rangle = 36$, and $\langle 3 \rangle = 5$. The permutation w avoids q.

3. Bruhat order

Standard terminology from the theory of partially ordered sets will be used throughout this paper. Good sources for information on this topic are [17] and [19].

The Bruhat order is a partial ordering that can be placed on a Coxeter group.

Definition 3.1. Fix a Coxeter group W generated by the simple reflections $\mathcal{S}$. Let $\overline{\mathcal{S}} = \{wsw^{-1}\colon\ w \in W,\ s \in \mathcal{S}\}$ be the set of reflections. For $w, w' \in W$, write $w \lessdot w'$ if $\ell(w') = \ell(w) + 1$ and $w' = \bar{s}w$ for some $\bar{s} \in \overline{\mathcal{S}}$. The covering relations $\lessdot$ give the (*strong*) *Bruhat order*.

As discussed in [1], the Bruhat order does not favor left or right multiplication, despite the appearance of Definition 3.1.

Example 3.2. $7314625 \gtrdot 7312645$. Although not a covering relation, $7314625 > 1374625$ in the Bruhat order.

This partial order has many properties which are discussed and proved in [1]. The property most relevant to this work, the subword property, gives an equivalent definition of the Bruhat order in terms of reduced decompositions.

Theorem *(Subword property). Let W be a Coxeter group and $w, w' \in W$. Choose $i_1 \cdots i_{\ell'} \in R(w')$. Then $w \leqslant w'$ in the Bruhat order if and only if there exists $j_1 \cdots j_\ell \in R(w)$ which is a subword of $i_1 \cdots i_{\ell'}$.*

The Bruhat order for a Coxeter group gives a graded poset where the rank function is the length of an element. Figures 1 and 2 give the Hasse diagrams for the Bruhat order on $\mathfrak{S}_4$ and $\mathfrak{S}_2^B$, respectively.

Other properties of the Bruhat order include that it is an Eulerian poset (shown by Verma in [20]) and that it is CL-shellable (shown by Björner and Wachs in [2]). Björner and Wachs also show that every open interval in the poset is topologically a sphere. For more information, see [2], [17], and [20].

Understanding intervals in the Bruhat order is made substantially simpler by Dyer's result in [5]: for any ℓ, there are only finitely many nonisomorphic intervals of length ℓ in the Bruhat

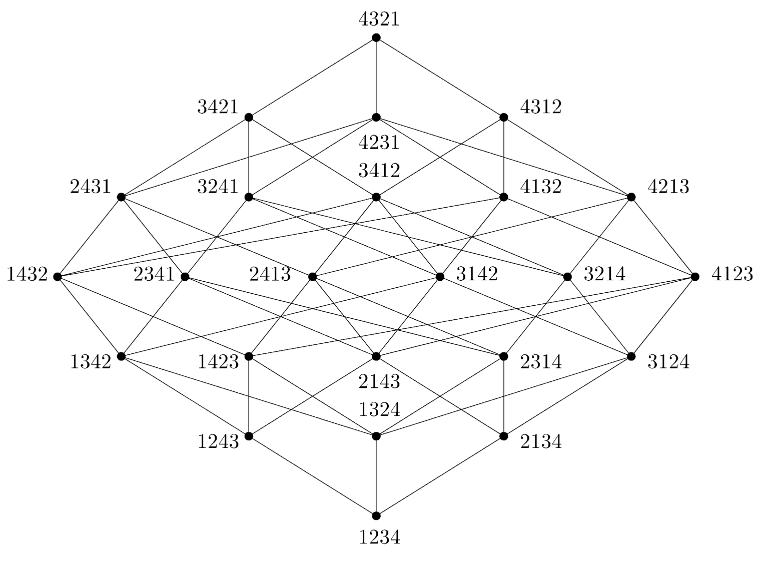

Fig. 1. The Bruhat order for $\mathfrak{S}_4$.

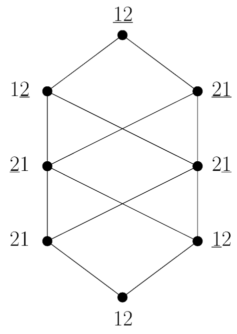

Fig. 2. The Bruhat order for $\mathfrak{S}_2^B$.

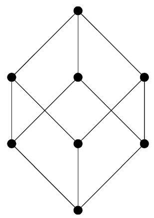

Fig. 3. The boolean poset B_3.

order of finite Coxeter groups. The length 4 intervals have been classified, as have length 5 intervals in $\mathfrak{S}_n$, by Hultman in [8,9].

Order ideals, specifically principal order ideals, are studied here.

Definition 3.3. Let W be a finite Coxeter group. For $w \in W$, let

$$B(w) = \{v \in W:\ v \leqslant w\}$$

be the principal order ideal of w in the Bruhat order for W.

In [13], Sjöstrand studies $B(w)$ in relation to rook configurations and Ferrers boards. He also gives a polynomial time recurrence for computing some $|B(w)|$.

4. Boolean principal order ideals

Definition 4.1. The *boolean poset* B_r is the set of subsets of $[r]$ ordered by set inclusion. A poset is *boolean* if it is isomorphic to B_r for some r.

Because $B_1 \cong B(21)$, a poset is boolean if and only if it is isomorphic to $B(21)^r$ for some r. For example, the poset depicted in Fig. 3 is isomorphic to $B(21)^3$. The goal of this section is to determine exactly when the principal order ideal $B(w)$ is boolean, for $w \in \mathfrak{S}_n$.

Definition 4.2. Let w be a permutation in $\mathfrak{S}_n$. If the poset $B(w)$ is boolean, then w is a *boolean* permutation.

Theorem 4.3. *The permutation w is boolean if and only if w is* 321- *and* 3412*-avoiding.*

Proof. Suppose that $B(w)$ is boolean. The poset $B(w)$ is graded of rank $\ell := \ell(w)$, and so $B(w) \cong B_\ell$. Fix $\boldsymbol{i} = i_1 \cdots i_\ell \in R(w)$. By the subword property, it must be possible to delete any subset of $\{i_1, \ldots, i_\ell\}$ and obtain a string which is still reduced. From this, it is straightforward to show that boolean permutations are exactly those which have a reduced decomposition with no repeated letters. In fact, if one reduced decomposition has this property, then all reduced decompositions do.

Suppose that w is not boolean. Then there is a reduced decomposition of w with a repeated letter. Therefore, there exists a reduced decomposition of w with one of the following factors, for some M:

$$(121)^M, \tag{6}$$

$$(2132)^M. \tag{7}$$

By the vexillary characterization of [18], a factor as in Eq. (6) indicates that w has a 321-pattern. Similarly, a factor as in Eq. (7) implies either a 3412-pattern, or a 4312-, 3421-, or 4321-pattern. The latter three all contain the pattern 321, so a repeated letter in a reduced decomposition implies that w has a 321- or a 3412-pattern.

Conversely, 321 and 3412 are vexillary. Thus, by the vexillary characterization, if either pattern appears then a reduced decomposition has a repeated letter. $\square$

Boolean permutations were previously enumerated by West in [21] and Fan in [7], although under different guises.

Corollary 4.4 *(Fan, West). The number of boolean permutations in $\mathfrak{S}_n$ is F_{2n-1}, where $\{F_0, F_1, \ldots\}$ are the Fibonacci numbers. This is sequence* A001519 *in* [14].

The boolean permutations can also be enumerated by length.

Corollary 4.5. *Let $L(n,k) := \#\{w \in \mathfrak{S}_n\colon\ \ell(w) = k \text{ and } w \text{ is boolean}\}$. Then*

$$L(n,k) = \sum_{i=1}^{k} \binom{n-i}{k+1-i}\binom{k-1}{i-1}, \tag{8}$$

where the (*empty*) *sum for $k = 0$ is defined to be* 1.

Proof. The result is proved by induction. First, observe that there is exactly one permutation in $\mathfrak{S}_n$ of length 0, and it is boolean. There are $n-1$ permutations in $\mathfrak{S}_n$ of length 1, and these are all boolean. Letting $k = 1$ in Eq. (8) yields

$$\sum_{i=1}^{1} \binom{n-i}{1+1-i}\binom{1-1}{i-1} = \binom{n-1}{1}\binom{0}{0} = n-1,$$

so the corollary holds for $k \leqslant 1$ and any $n > k$.

Assume the result for all $k \in [0, K)$ and $n \in [1, N)$. A boolean permutation avoids the patterns 321 and 3412. Suppose $w \in \mathfrak{S}_N$ is a boolean permutation with $\ell(w) = K$, and consider the location of N in the one-line notation of w.

- If $w(N) = N$, then $w(1) \cdots w(N-1) \in \mathfrak{S}_{N-1}$ can be any boolean permutation of length K.

- If $w(N-1)=N$, then $w(1)\cdots w(N-2)w(N)\in\mathfrak{S}_{N-1}$ can be any boolean permutation of length $K-1$.
- If $w(N-2)=N$, then $w(N)=N-1$. Thus, $w(1)\cdots w(N-3)w(N-1)\in\mathfrak{S}_{N-2}$ can be any boolean permutation of length $K-2$.
- If $w(N-3)=N$, then $w(N)=N-1$ and $w(N-1)=N-2$. Therefore $w(1)\cdots w(N-4)w(N-2)\in\mathfrak{S}_{N-3}$ can be any boolean permutation of length $K-3$.
- $\ldots$.

Thus $L(N,K)=L(N-1,K)+\sum_{i=1}^{K}L(N-i,K-i)$. Combining this with the analogous equation for $L(N-1,K-1)$ yields

$$\begin{aligned}L(N,K)&=L(N-1,K-1)+L(N-1,K-1)+L(N-2,K-1)+\cdots\\&\quad+L(K,K-1).\end{aligned}$$

The fact that w is boolean implies that $K\leqslant N$. Therefore, all of the terms in the above equations are well defined.

By the inductive assumptions and basic facts about binomial coefficients,

$$\begin{aligned}L(N,K)&=\sum_{i=1}^{K-1}\binom{N-1-i}{K-i}\binom{K-2}{i-1}+\sum_{j=K}^{N-1}\sum_{i=1}^{K-1}\binom{j-i}{K-i}\binom{K-2}{i-1}\\&=\sum_{i=1}^{K-1}\binom{N-1-i}{K-i}\binom{K-2}{i-1}+\sum_{i=1}^{K-1}\binom{N-i}{K+1-i}\binom{K-2}{i-1}\\&=\sum_{i=2}^{K}\binom{N-i}{K+1-i}\binom{K-2}{i-2}+\sum_{i=1}^{K-1}\binom{N-i}{K+1-i}\binom{K-2}{i-1}\\&=\binom{N-K}{1}\binom{K-2}{K-2}+\sum_{i=2}^{K-1}\binom{N-i}{K+1-i}\left(\binom{K-2}{i-2}+\binom{K-2}{i-1}\right)\\&\quad+\binom{N-1}{K}\binom{K-2}{0}\\&=\sum_{i=1}^{K}\binom{N-i}{K+1-i}\binom{K-1}{i-1}.\qquad\Box\end{aligned}$$

The numbers $L(n,k)$ are equal to the numbers $T(n,n-k)$ in sequence A105306 of [14]. From this, it is straightforward to compute the generating function

$$\sum_{n,k}L(n,k)t^kz^n=\frac{z(1-zt)}{1-2zt-z(1-zt)}.\tag{9}$$

For small n and k, the values $L(n,k)$ are displayed in Table 1.

It is interesting to note that $B(w)$ is boolean if and only if it is a lattice. The ideal $B(w)$ is a lattice if and only if all of the R-polynomials are of the form $(q-1)^{\ell(y)-\ell(x)}$, as discussed by Brenti in [3]. Moreover, Brenti shows that this is equivalent to all of the Kazhdan–Lusztig polynomials equaling the g-polynomials of the duals of the corresponding subintervals. The g-polynomials are defined in [15], and their coefficients are the toric g-vectors.

Table 1
The number of boolean permutations of each length in $\mathfrak{S}_1, \ldots, \mathfrak{S}_8$

$L(n,k)$	$k=0$	1	2	3	4	5	6	7
$n=1$	1							
2	1	1						
3	1	2	2					
4	1	3	5	4				
5	1	4	9	12	8			
6	1	5	14	25	28	16		
7	1	6	20	44	66	64	32	
8	1	7	27	70	129	168	144	64

Missing table entries are equal to 0.

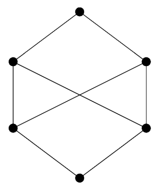

Fig. 4. The principal order ideal $B(321)$.

Subsequent to Theorem 4.3, it is natural to ask the following questions. What can be said about the principal order ideal of permutations with exactly one occurrence of exactly one of the patterns 321 or 3412? (Note that these have reduced decompositions in which exactly one letter is repeated, and it appears exactly twice.) In particular, how big are these ideals? These questions are answered below. Generalizations allowing more occurrences of 321 and 3412 are not treated here.

- Suppose that w has exactly one 321-pattern and is 3412-avoiding. Let $\ell = \ell(w)$. There exists $i_1 \cdots i_\ell \in R(w)$ such that $i_j = i_{j+2}$ for some j, and there are no repeated letters besides i_j and i_{j+2}. The subword property dictates the poset $B(w)$ as follows. Consider the poset B_ℓ of subsets of $[\ell]$ ordered by set inclusion. Delete all elements of the poset containing $\{j, j+2\}$ but not $j+1$, and identify all elements of the poset containing j but not $\{j+1, j+2\}$ with those that interchange the roles of $j+2$ and j. The resulting poset is isomorphic to $B(w)$, and $|B(w)| = 3 \cdot 2^{\ell-2}$.
  Figure 4 depicts $B(w)$ for the simplest such permutation, $w = 321$.
- Suppose that w has exactly one 3412-pattern and is 321-avoiding. Let $\ell = \ell(w)$. There exists $i_1 \cdots i_\ell \in R(w)$ such that $i_j i_{j+1} i_{j+2} i_{j+3} = (2132)^M$ for some M, and there are no repeated letters besides i_j and i_{j+3}. Again, consider the poset B_ℓ of subsets of $[\ell]$ ordered by set inclusion. Delete all elements of the poset containing $\{j, j+3\}$ but not $\{j+1, j+2\}$, and identify those elements of the poset that contain j but not $\{j+1, j+2, j+3\}$ with those that interchange the roles of $j+3$ and j. The resulting poset is isomorphic to $B(w)$, and $|B(w)| = 7 \cdot 2^{\ell-3}$.
  Figure 5 depicts $B(w)$ for the simplest such permutation, $w = 3412$.

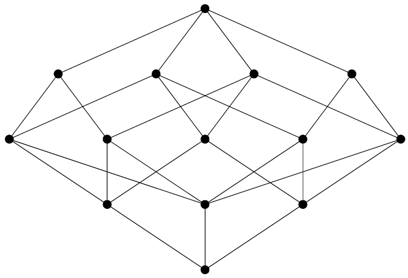

Fig. 5. The principal order ideal $B(3412)$.

5. Principal order ideals isomorphic to a power of $B(w_0^{(k)})$

The previous section characterized all permutations for which $B(w)$ is boolean, where a boolean poset is one which is isomorphic to some power of $B(21)$. This section generalizes the previous work by describing the permutations for which $B(w)$ is isomorphic to a power of $B(w_0^{(k)})$ for $k \geqslant 3$, where $w_0^{(k)}$ is the longest element in $\mathfrak{S}_k$.

Definition 5.1. Let $k \geqslant 3$ be an integer and $w \in \mathfrak{S}_n$ be a permutation. If $B(w) \cong B(w_0^{(k)})^r$ for some r, then w is a *power permutation*.

As in the previous section, the characterization of power permutations is in terms of patterns, although not in quite the same way as Theorem 4.3. A few preliminaries are necessary before this characterization can be stated.

Proposition 5.2. *For $x, y \in \mathfrak{S}_n$, suppose that $[x, y] \cong B(w_0^{(k)})$ for some k. Then there exist $\boldsymbol{i} \in R(x)$ and $\boldsymbol{j} \in R(y)$ such that $\boldsymbol{i}$ is obtained by deleting a factor from $\boldsymbol{j}$ which is the shift of an element of $R(w_0^{(k)})$.*

Proof. Let $\boldsymbol{i} \in R(x)$ and $\boldsymbol{j} \in R(y)$ be, by the subword property, such that $\boldsymbol{i}$ is a subword of $\boldsymbol{j}$. Consider the multiset S of $\binom{k}{2}$ letters deleted from $\boldsymbol{j}$ to form $\boldsymbol{i}$. Because $[x, y] \cong B(w_0^{(k)})$, this S contains $k-1$ distinct letters.

The number of distinct letters in S equals the number of elements covering x in $[x, y]$. Therefore it must be possible to find $\boldsymbol{i}$ and $\boldsymbol{j}$ as above so that the factors in $\boldsymbol{j}$ formed by S have the property that equal elements of S lie in the same factor.

Given T distinct and consecutive letters, the longest reduced decomposition that can be formed by them has length $\binom{T+1}{2}$. Observe that

$$\binom{T_1+1}{2} + \binom{T_2+1}{2} < \binom{T_1+T_2+1}{2}$$

for $T_1, T_2 > 0$. Thus, all of S comprises a single factor in $\boldsymbol{j}$. $\square$

Proposition 5.3. *If $k \geqslant 4$, and $\boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_r} \in R(w)$ for $\boldsymbol{j} \in R(w_0^{(k)})$, then the M_is are distinct.*

Proof. Fix $k \geqslant 4$. It is straightforward to show that if $\boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_r} \in R(w)$, then

$$w(x + M_i) = k + 1 - x + M_i \quad \text{for } x \in [2, k-1], \tag{10}$$

for all $i \in [r]$. The result follows from Eq. (10) and the fact that elements of $R(w)$ are reduced. $\square$

The result does not hold for $k = 3$ because Eq. (10) says only that $2 + M_i$ is fixed by w for all i. For example, $(121)^2(121)^0(121)^4(121)^2 \in R(5274163)$.

Definition 5.4. An inversion (i, j) is *in a p-pattern* if there is an occurrence $\langle p\rangle$ in w such that $\langle p(i')\rangle = w(i)$ and $\langle p(j')\rangle = w(j)$ for some i' and j'.

Proposition 5.5. *Fix $k \geqslant 3$. Every inversion in w is in exactly one $w_0^{(k)}$-pattern if and only if there exists*

$$\boldsymbol{i} = \boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_r} \in R(w) \tag{11}$$

for $\boldsymbol{j} \in R(w_0^{(k)})$, where the M_is are distinct. (Consequently there are r occurrences of the pattern $w_0^{(k)}$ in w.) Because $\boldsymbol{i}$ is reduced, $|M_i - M_j| \geqslant k - 1$ for all $i \neq j$.

Proof. Fix $k \geqslant 3$. The result is straightforward for permutations with zero or one $w_0^{(k)}$-pattern. Suppose that $w \in \mathfrak{S}_n$ has $r > 1$ occurrences of $w_0^{(k)}$, and that every inversion in w is in exactly one $w_0^{(k)}$-pattern. At least one of these patterns occurs in consecutive positions. Therefore, for some M, there exists

$$w' := w \cdot \big(1 \cdots M(k+M)(k-1+M)\cdots(1+M)(k+1+M)\cdots n\big)$$

where every inversion in w' is in exactly one $w_0^{(k)}$-pattern, and there are $r-1$ such patterns. Thus, by induction, there exists $\boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_r} \in R(w)$ for $\boldsymbol{j} \in R(w_0^{(k)})$. If $k > 3$, then this direction of the proof is complete by Proposition 5.3. If $k = 3$ and the M_is are not distinct, then the permutation w would necessarily have a 4312-, 4231-, or 3421-pattern, which contradicts the original hypothesis.

Conversely, suppose that w has a reduced decomposition as in Eq. (11), where the M_is are distinct. Consider applying the braid relations to $\boldsymbol{i}$. It is impossible to get a factor equal to the shift of an element of $R(w_0^{(k+1)})$. Likewise any shift of an element of $R(4312)$, $R(4231)$, or $R(3421)$ can be extended to a factor that is the shift of an element of $R(4321)$. Therefore the vexillary characterization implies that w avoids $w_0^{(k+1)}$, 4312, 4231, and 3421. Consequently, every inversion in w is in exactly one $w_0^{(k)}$-pattern. $\square$

With this groundwork, the main theorem of the section can now be stated.

Theorem 5.6. *The permutation w is a power permutation, namely $B(w)$ is a power of $B(w_0^{(k)})$, if and only if every inversion in w is in exactly one $w_0^{(k)}$-pattern for some fixed $k \geqslant 3$.*

Proof. Fix $k \geqslant 3$ and $w \in \mathfrak{S}_n$. First suppose that every inversion in w is in exactly one $w_0^{(k)}$-pattern, and that w contains R distinct occurrences of the pattern $w_0^{(k)}$. "Undoing" inversions in one of these patterns does not alter the other patterns. Consequently $B(w) \cong B(w_0^{(k)})^R$.

For the other direction of the proof, suppose that $B(w) \cong B(w_0^{(k)})^R$, and proceed by induction on R. The case $R = 0$ is trivial, and the case $R = 1$ was considered in Proposition 5.2. Suppose that the theorem holds for permutations whose principal order ideals are isomorphic to $B(w_0^{(k)})^r$, for all $r \in [0, R)$.

There are distinct permutations $w_1, \ldots, w_R$, each less than w, with

$$B(w_1) \cong \cdots \cong B(w_R) \cong B\big(w_0^{(k)}\big)^{R-1}.$$

By Proposition 5.5 and the inductive hypothesis, each of these R permutations has a reduced decomposition $\boldsymbol{j}^{M_1^h} \cdots \boldsymbol{j}^{M_{R-1}^h} \in R(w_h)$, where the M_i^hs are distinct.

The interval $[w_h, w]$ is isomorphic to $B(w_0^{(k)})$ for all h, so Proposition 5.2 indicates that w has a reduced decomposition $\boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_R}$. The distinct permutations $w_1, \ldots, w_R$ each satisfy the induction hypothesis. Hence the M_is are distinct, and Proposition 5.5 completes the proof. □

The following corollary is stated in the language of [18], where $X(w)$ is Elnitsky's polygon (defined in [6]) and all the polygons have unit sides.

Corollary 5.7. *If w is a power permutation, then there is a zonotopal tiling of $X(w)$ consisting entirely of $2k$-gons for some $k \geqslant 3$. The converse is true if $k \geqslant 4$.*

Theorem 5.6 gives a concise description of power permutations, again in terms of patterns. Although the flavor of this description differs from that of Theorem 4.3, the prominent role of patterns in the power permutation characterization is immediately apparent. It is clear that Theorem 5.6 must be restricted to $k \geqslant 3$, while the $k = 2$ case is treated in Theorem 4.3, because to say that "every inversion in w is in exactly one 21-pattern" provides no information.

It is instructive to consider what it means for every inversion in $w \in \mathfrak{S}_n$ to be in exactly one $w_0^{(k)}$-pattern. The following facts are straightforward to show.

- Distinct occurrences of $w_0^{(k)}$ are disjoint or share exactly one entry.
- If two occurrences of $w_0^{(k)}$ intersect, then either $\langle k\rangle_1 = \langle k\rangle_2$ or $\langle 1\rangle_1 = \langle 1\rangle_2$.
- Without loss of generality, all values in $\langle w_0^{(k)}\rangle_1$ are at least as large as all values in $\langle w_0^{(k)}\rangle_2$. The nonshared values in $\langle w_0^{(k)}\rangle_1$ all occur to the right of the nonshared values in $\langle w_0^{(k)}\rangle_2$.
- If m is not in any $w_0^{(k)}$-pattern, then $w(m) = m$ and m is not in any element of $R(w)$. Also, $w(1)\cdots w(m-1) \in \mathfrak{S}_{m-1}$ and $(w(m+1)-m)\cdots(w(n)-m) \in \mathfrak{S}_{n-m}$ are power permutations with the same parameter k.
- The values $\langle k-1\rangle, \ldots, \langle 2\rangle$ occur consecutively in w.

Example 5.8. $521436 \in \mathfrak{S}_6$ and $432159876 \in \mathfrak{S}_9$ are both power permutations.

6. Patterns and order ideals

The previous sections considered principal order ideals in the Bruhat order of the symmetric group. This section examines order ideals that are not necessarily principal. The following questions are completely answered.

(1) For what $p \in \mathfrak{S}_k$, where $k \geqslant 3$, is the set

$$S_n\{p\} = \{w \in \mathfrak{S}_n\colon\ w \text{ is } p\text{-avoiding}\}$$

a nonempty order ideal, for some $n > k$?

(2) For what $p \in \mathfrak{S}_k$ and $q \in \mathfrak{S}_l$, where $k, l \geqslant 3$, is the set

$$S_n\{p,q\} = \{w \in \mathfrak{S}_n\colon\ w \text{ is } p\text{- and } q\text{-avoiding}\}$$

a nonempty order ideal for some $n \geqslant k, l$?

The restrictions on n, k, and l eliminate trivial cases.

Somewhat surprisingly, very few patterns answer the above questions.

Theorem 6.1. *For $k \geqslant 3$, there is no permutation $p \in \mathfrak{S}_k$ for which there exists $n > k$ such that the set $S_n\{p\}$ is an order ideal.*

Proof. If $w_0^{(n)}$ avoids p, then $S_n\{p\}$ is not an order ideal: $p(1)\cdots p(k)(k+1)\cdots n$ is less than $w_0^{(n)}$ and not in $S_n\{p\}$. Thus $p = w_0^{(k)}$.

Let $w = k(k+1)(k-1)(k-2)\cdots 4312(k+2)(k+3)\cdots n \in S_n\{p\}$. Then

$$\begin{aligned} w &> k(k+1)(k-1)(k-2)\cdots 4132(k+2)(k+3)\cdots n \\ &> k(k+1)(k-1)(k-2)\cdots 1432(k+2)(k+3)\cdots n \\ &> \cdots \\ &> k(k+1)1(k-1)(k-2)\cdots 432(k+2)(k+3)\cdots n \\ &> 1(k+1)k(k-1)(k-2)\cdots 432(k+2)(k+3)\cdots n =: v. \end{aligned}$$

Because v has a p-pattern, the set $S_n\{p\}$ is not an order ideal for any $n > k$. □

The set $S_n\{321, 3412\}$ of boolean permutations is an order ideal. Thus there are permutations p and q for which the set $S_n\{p,q\}$ is an order ideal.

Theorem 6.2. *Let $p \in \mathfrak{S}_k$ and $q \in \mathfrak{S}_l$ for $k, l \geqslant 3$. The only times when $S_n\{p,q\}$ is a nonempty order ideal for some $n \geqslant k, l$ are* $S_n\{321, 3412\}$, $S_n\{321, 231\}$, *and* $S_n\{321, 312\}$. *These sets are order ideals for all $n \geqslant 4$.*

Proof. As in the previous proof, it can be assumed that $p = w_0^{(k)}$.

Suppose that $S_n\{p,q\}$ is a nonempty order ideal for some $n \geqslant k, l$. Then the permutations below cannot be in $S_n\{p,q\}$, because they are all larger in the Bruhat order than a permutation containing a p-pattern:

$$\begin{aligned} &k\cdots 3(k+1)12(k+2)\cdots n, \\ &k(k+1)1(k-1)\cdots 32(k+2)\cdots n, \\ &1\cdots(n-k-1)(n-1)\cdots(n-k+2)n(n-k)(n-k+1), \\ &1\cdots(n-k-1)(n-1)n(n-k)(n-2)\cdots(n-k+2)(n-k+1). \end{aligned}$$

These all avoid p, so they must contain q. The only patterns in all of these permutations are $\{312, 231, 3412, 12\cdots(l-1)l\}$. If $q = 12\cdots(l-1)l$ and $S_n\{p,q\}$ is nonempty, then it is not an order ideal because every element in $S_n\{p,q\}$ is greater than $12\cdots n \notin S_n\{p,q\}$. Therefore $q \in \{312, 231, 3412\}$.

Suppose that $k > 3$. If $q \in \{231, 312\}$, then $u = 32145 \cdots n \in S_n\{p, q\}$. However, $u > q(1)q(2)q(3)45 \cdots n \notin S_n\{p, q\}$. Similarly, $v = 342156 \cdots n \in S_n\{p, 3412\}$, but $v > 341256 \cdots n \notin S_n\{p, 3412\}$. Thus $k = 3$ if $S_n\{p, q\}$ is to be an order ideal.

By the vexillary characterization and Theorem 4.3, the set $S_n\{321, 231\}$ consists of those permutations that have reduced decompositions $i_1 \cdots i_\ell$ for $i_1 > \cdots > i_\ell$. Thus $S_n\{321, 231\}$ is an order ideal by the subword property. Similarly, the set $S_n\{321, 312\}$ consists of those permutations that have reduced decompositions $i_1 \cdots i_\ell$ for $i_1 < \cdots < i_\ell$. Once again, this is an order ideal. As stated earlier, the set $S_n\{321, 3412\}$ of boolean permutations is also an order ideal. □

The elements of $S_n\{321, 3412\}$ were enumerated by length in Corollary 4.5, and their rank generating function is Eq. (9). The enumerations for the sets $S_n\{321, 231\}$ and $S_n\{321, 312\}$ are straightforward.

Corollary 6.3. *The number of elements of length k in each of $S_n\{321, 231\}$ and $S_n\{321, 312\}$ is $\binom{n-1}{k}$. Consequently, each has rank generating function*

$$\sum_{n,k} \binom{n-1}{k} t^k z^n = \frac{z}{1-(1+t)z}.$$

Proof. A length k element in $S_n\{321, 231\}$ has a reduced decomposition $i_1 \cdots i_k$ where $i_1 > \cdots > i_k$. Therefore, it is uniquely determined by choosing k of the $n-1$ possible letters. The enumeration for $S_n\{321, 312\}$ is analogous. □

In each instance where $S_n\{p, q\}$ is an order ideal, the rank generating function of this subposet is a rational function. For $S_n\{321, 231\}$ and $S_n\{321, 312\}$, these order ideals are actually *principal*: $S_n\{321, 231\} = B(n12 \cdots (n-1))$, and $S_n\{321, 312\} = B(23 \cdots n1)$. Results of Lakshmibai and Sandhya (see [11]) and Carrell and Peterson (see [4]) show that $B(w)$ is rank symmetric if and only if w is 3412- and 4231-avoiding, which shows (although it is already clear from Corollary 6.3) that $S_n\{321, 231\}$ and $S_n\{321, 312\}$ are both rank symmetric.

The poset of boolean permutations, $S_n\{321, 3412\}$, is simplicial, and its f-vector was computed in Eq. (8). In [16], Stanley showed that for a given vector $\boldsymbol{h}$, there exists a Cohen–Macaulay simplicial poset with h-vector equal to $\boldsymbol{h}$ if and only if $h_0 = 1$ and $h_i \geqslant 0$ for all i. The last coordinate of the h-vector of $S_n\{321, 3412\}$ is $L(n, n-1) - L(n, n-2)$, which is negative for $n > 3$ (the only n for which $S_n\{321, 3412\}$ is defined). Thus $S_n\{321, 3412\}$ is never Cohen–Macaulay.

7. Boolean order ideals in the Bruhat order for types B and D

As Section 4 studied boolean principal order ideals in $\mathfrak{S}_n$, this section does likewise for signed permutations. Recall that the finite Coxeter groups of types B and D consist of signed permutations, where $\mathfrak{S}_n^D \subset \mathfrak{S}_n^B$ is the subset of elements that have an even number of negative signs when written in one-line notation.

Example 7.1. $\mathfrak{S}_2^B = \{12, 21, \underline{1}2, \underline{2}1, 1\underline{2}, 2\underline{1}, \underline{1}\underline{2}, \underline{2}\underline{1}\}$ and $\mathfrak{S}_2^D = \{12, 21, \underline{1}\underline{2}, \underline{2}\underline{1}\}$.

The central object here is the principal order ideal of a signed permutation.

Definition 7.2. Let W be a finite Coxeter group of type A, B, or D. The element $w \in W$ is *boolean* if $B(w)$ is a boolean poset.

The following proposition holds for $\mathfrak{S}_n^B$ and $\mathfrak{S}_n^D$ as well the symmetric group, and its proof is omitted.

Proposition 7.3. *Let W be a finite Coxeter group of type A, B, or D. An element $w \in W$ is boolean if and only if a reduced decomposition of w has no repeated letters.*

Proposition 7.3 resembles a result of Fan in [7] for an arbitrary Weyl group W. Fan showed that if the reduced decompositions of $w \in W$ avoid factors of the form sts, then the corresponding Schubert variety X_w is smooth if and only if some (every) reduced decomposition of w contains no repeated letter.

The classifications of the boolean elements in $\mathfrak{S}_n^B$ and $\mathfrak{S}_n^D$ rely on Proposition 7.3. In each case, the boolean elements are described and enumerated by length. As with $\mathfrak{S}_n$, these characterizations are in terms of patterns, although the type B case is more complicated than type A, and type D is more complicated still.

Theorem 7.4. *The signed permutation $w \in \mathfrak{S}_n^B$ is boolean if and only if w avoids all of the following patterns.*

$\underline{12}$	$\underline{21}$
321	3412
$32\underline{1}$	$34\underline{1}2$
$\underline{3}21$	$\underline{3}412$
$1\underline{2}$	$3\underline{2}1$

Proof. By Proposition 7.3, a reduced decomposition of a boolean element contains at most one 0. Therefore boolean elements in $\mathfrak{S}_n^B$ have at most one negative value. Thus the patterns $\underline{12}$ and $\underline{21}$ must be avoided. Similarly, $w \in \mathfrak{S}_n^B$ is boolean if and only if it has a reduced decomposition with one of the following forms:

(1) an ordered subset of $[n-1]$;
(2) 0 {an ordered subset of $[n-1]$}; or
(3) {an ordered subset of $[n-1]$} 0.

By Theorem 4.3, a reduced decomposition of $v \in \mathfrak{S}_n$ is an ordered subset of $[n-1]$ if and only if v is 321- and 3412-avoiding. The product $s_0 v$ changes the sign of the value 1, while $v s_0$ changes the sign of the value in the first position. Therefore, a boolean permutation in $\mathfrak{S}_n^B$ also avoids $32\underline{1}$, $34\underline{1}2$, $\underline{3}21$, and $\underline{3}412$.

Finally, a negative value can appear in a boolean permutation in $\mathfrak{S}_n^B$ only if it is $\underline{1}$ or occurs in the first position. Thus the permutation also avoids $1\underline{2}$ and $3\underline{2}1$. $\square$

Proposition 7.3 states that $w \in \mathfrak{S}_n^B$ is boolean if and only if it has a reduced decomposition whose letters are all distinct. Given previous results, the enumeration of these elements is straightforward. Each of $\{0, 1, \ldots, n-1\}$ can appear at most once in a reduced decomposition of a boolean element, so it is necessary only to understand when two ordered subsets of $\{0, 1, \ldots, n-1\}$ correspond to the same permutation. There is a bijection between pairs of commuting elements in $\{s_0, s_1, \ldots, s_{n-1}\}$ and pairs of commuting elements in $\{s_1, \ldots, s_{n-1}, s_n\}$. Therefore, the work of enumerating boolean elements in $\mathfrak{S}_n^B$ was already done in Section 4.

Corollary 7.5. *The number of boolean signed permutations in* $\mathfrak{S}_n^B$ *is* F_{2n+1}.

Proof. The number of boolean signed permutations in $\mathfrak{S}_n^B$ is equal to the number of boolean unsigned permutations in $\mathfrak{S}_{n+1}$, which is F_{2n+1} by Corollary 4.4. $\square$

The previous result was also obtained by Fan in [7].

Corollary 7.6. *The number of boolean signed permutations in* $\mathfrak{S}_n^B$ *of length k is*

$$\sum_{i=1}^{k} \binom{n+1-i}{k+1-i} \binom{k-1}{i-1},$$

where the (*empty*) *sum for* $k=0$ *is defined to be* 1.

Proof. The number of boolean signed permutations in $\mathfrak{S}_n^B$ of length k is equal to the number of boolean unsigned permutations in $\mathfrak{S}_{n+1}$ of length k. $\square$

The boolean elements of $\mathfrak{S}_n^D$ are defined and enumerated below. As for types A and B, this characterization is in terms of patterns avoidance.

Theorem 7.7. *The signed permutation* $w \in \mathfrak{S}_n^D$ *is boolean if and only if* w *avoids all of the following patterns.*

$$\begin{array}{llllll}
\underline{123} & \underline{132} & \underline{213} & \underline{231} & \underline{312} & \underline{321} \\
321 & 3412 & & & & \\
32\underline{1} & 3\underline{12} & 34\underline{1}2 & 34\underline{21} & & \\
\underline{3}21 & \underline{23}1 & \underline{3}412 & \underline{43}12 & & \\
1\underline{2} & 3\underline{2}1 & & & & \\
\underline{3}2\underline{1} & \underline{3}4\underline{1}2 & & & &
\end{array}$$

Note that some of these patterns have an odd number of negative values.

Proof. By Proposition 7.3, a reduced decomposition of a boolean element has at most one $1'$. Therefore boolean elements in $\mathfrak{S}_n^D$ have at most two negative values, so $\underline{123}$, $\underline{132}$, $\underline{213}$, $\underline{231}$, $\underline{312}$, and $\underline{321}$ must be avoided. Similarly, $w \in \mathfrak{S}_n^D$ is boolean if and only if it has a reduced decomposition with one of the following forms:

(1) an ordered subset of $[n-1]$;
(2) $1'$ {an ordered subset of $[n-1]$}; or
(3) {an ordered subset of $[n-1]$} $1'$.

By Theorem 4.3, a reduced decomposition of $v \in \mathfrak{S}_n$ is an ordered subset of $[n-1]$ if and only if v is 321- and 3412-avoiding. The product $s_{1'}v$ maps the value 1 to $\underline{2}$ and the value 2 to $\underline{1}$, while $vs_{1'} = \underline{v(2)v(1)}v(3)\cdots v(n)$. Therefore, a boolean permutation in $\mathfrak{S}_n^D$ also avoids $32\underline{1}$, $3\underline{12}$, $34\underline{1}2$, $34\underline{21}$, $\underline{3}21$, $\underline{23}1$, $\underline{3}412$, and $\underline{43}12$.

Finally, since negative values in a boolean permutation in $\mathfrak{S}_n^D$ can only appear either as $\underline{1}$ and $\underline{2}$ or in the first two positions, the permutation must also avoid the patterns $1\underline{2}$, $3\underline{2}1$, $\underline{3}2\underline{1}$, and $\underline{3}4\underline{1}2$. $\square$

Table 2
The number of boolean elements of each length in $\mathfrak{S}_1^D, \ldots, \mathfrak{S}_8^D$

$L^D(n,k)$	$k=0$	1	2	3	4	5	6	7	8
$n=1$	1	0							
2	1	2	1						
3	1	3	5	4					
4	1	4	9	13	8				
5	1	5	14	26	30	16			
6	1	6	20	45	69	68	32		
7	1	7	27	71	133	176	152	64	
8	1	8	35	105	230	373	436	336	128

Missing table entries are equal to 0.

As in types A and B, the boolean elements in type D can be enumerated, although this enumeration is not as simple to state as in the other types. Fan computed these values in [7], with the following results.

Corollary 7.8 *(Fan). For $n \geqslant 4$, the number of boolean elements in $\mathfrak{S}_n^D$ is*

$$\frac{13-4b}{a^2(a-b)}a^n + \frac{13-4a}{b^2(b-a)}b^n,$$

where $a = (3+\sqrt{5})/2$ and $b = (3-\sqrt{5})/2$.

Corollary 7.9. *For $n > 1$, the number of boolean elements in $\mathfrak{S}_n^D$ of length k is*

$$L^D(n,k) := L(n,k) + 2L(n,k-1) - L(n-2,k-1) - L(n-2,k-2), \tag{12}$$

where $L(n,k)$ is as defined previously, and $L(n,k)$ is 0 for any (n,k) on which it is undefined. $L^D(1,0) = 1$ and $L^D(1,1) = 0$.

Proof. These enumerative results follow from Theorem 7.7 and Corollary 4.5. The subtracted terms in Eq. (12) resolve the overcounting that occurs when the reduced decompositions of a boolean element contain $1'$ but not 2. The case $n = 1$ must be treated separately because the only element in $\mathfrak{S}_1^D$ is the identity. □

For small n and k, the values $L^D(n,k)$ are displayed in Table 2.

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