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Bruhat order, smooth Schubert varieties, and hyperplane arrangements ☆

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Abstract

The aim of this article is to link Schubert varieties in the flag manifold with hyperplane arrangements. For a permutation, we construct a certain graphical hyperplane arrangement. We show that the generating function for regions of this arrangement coincides with the Poincaré polynomial of the corresponding Schubert variety if and only if the Schubert variety is smooth. We give an explicit combinatorial formula for the Poincaré polynomial. Our main technical tools are chordal graphs and perfect elimination orderings.

Keywords: Flag manifold; Schubert varieties; Smoothness; Hyperplane arrangements; Bruhat order; Poincaré polynomials; Chordal graphs; Perfect elimination ordering

1. Introduction

For a permutation $w \in S_n$, let $P_w(q) := \sum_{u \leqslant w} q^{\ell(u)}$, where the sum is over all permutations $u \in S_n$ below w in the strong Bruhat order. Geometrically, the polynomial $P_w(q)$ is the Poincaré polynomial of the Schubert variety $X_w = BwB/B$ in the flag manifold $SL(n, \mathbb{C})/B$.

Define the *inversion hyperplane arrangement* $\mathcal{A}_w$ as the collection of the hyperplanes $x_i - x_j = 0$ in $\mathbb{R}^n$, for all inversions $1 \leqslant i < j \leqslant n$, $w(i) > w(j)$. Let $R_w(q) := \sum_r q^{d(r_0,r)}$ be the generating function that counts regions r of the arrangement $\mathcal{A}_w$ according to the distance $d(r_0, r)$ from the fixed initial region r_0 such that $(1, \ldots, n) \in r_0$.

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The main result of the paper is the claim that $P_w(q) = R_w(q)$ if and only if the Schubert variety X_w is smooth.

According to the well-known Lakshmibai–Sandhya's criterion [10], the Schubert variety X_w is smooth if and only if the permutation w avoids two patterns 3412 and 4213. (Let us say that the permutation w is smooth in this case.) Also Carrell–Peterson [4] proved that X_w is smooth if and only if the Poincaré polynomial $P_w(q)$ is palindromic, that is $P_w(q) = q^{\ell(w)} P_w(q^{-1})$. If w is not smooth, then the polynomial $P_w(q)$ is not palindromic, but the polynomial $R_w(q)$ is always palindromic. So $P_w(q) \neq R_w(q)$ in this case. On the other hand, we show that, for smooth w, the polynomials $R_w(q)$ and $P_w(q)$ satisfy the same recurrence relation. For the Poincaré polynomials $P_w(q)$, this recurrence relation was given by Gasharov [6]. This implies that $P_w(q) = R_w(q)$ in this case.

For smooth w, we present an explicit factorization of the polynomials $P_w(q) = R_w(q)$ as a product of q-numbers $[e_1 + 1]_q \cdots [e_n + 1]_q$, where $e_1, \dots, e_n$ can be computed using the left-to-right maxima (aka records) of the permutation w. In this case, the *inversion graph* G_w, whose edges correspond to inversions in w, is a chordal graph. The numbers $e_1, \dots, e_n$ are the roots of the chromatic polynomial $\chi_{G_w}(t)$ of the inversion graph. The polynomial $\chi_{G_w}(t)$ is also the characteristic polynomial of the inversion hyperplane arrangement $\mathcal{A}_w$. We call the numbers $e_1, \dots, e_n$ the *exponents.*

2. Bruhat order and Poincaré polynomials

The (strong) *Bruhat order* "$\leqslant$" on the *symmetric group* S_n is the partial order generated by the relations $w < w \cdot t_{ij}$ if $\ell(w) < \ell(w \cdot t_{ij})$. Here $t_{ij} \in S_n$ is the transposition of i and j; and $\ell(w)$ denotes the *length* of a permutation $w \in S_n$, i.e., the number of inversions in w.

Intervals in the Bruhat order play a role in Schubert calculus and in Kazhdan–Lusztig theory. In this paper we concentrate on Bruhat intervals of the form $[\mathrm{id}, w] := \{u \in S_n \mid u \leqslant w\}$ (where $\mathrm{id} \in S_n$ is the identity permutation), that is, on lower order ideals of the Bruhat order. They are related to *Schubert varieties* $X_w = BwB/B$ in the flag manifold $SL(n, \mathbb{C})/B$. Here B denotes the Borel subgroup of $SL(n, \mathbb{C})$. The *Poincaré polynomial* of the Schubert variety X_w is the rank generating function for the interval $[\mathrm{id}, w]$, e.g., see [2]

$$P_w(q) = \sum_{u \leqslant w} q^{\ell(u)}.$$

The well-known smoothness criterion for Schubert varieties, due to Lakshmibai and Sandhya, is based on pattern avoidance. A permutation $w \in S_n$ contains a *pattern* $\sigma \in S_k$ if there is a subword with k letters in w with the same relative order of the letters as in the permutation σ. A permutation w *avoids the pattern* σ if w does not contain this pattern.

Theorem 1. *(See Lakshmibai and Sandhya [10].) For a permutation $w \in S_n$, the Schubert variety X_w is smooth if and only if w avoids the two patterns* 3412 *and* 4231.

We will say that $w \in S_n$ is a *smooth permutation* if it avoids these two patterns 3412 and 4231.

Another smoothness criterion, due to Carrell and Peterson, is given in terms of the Poincaré polynomial $P_w(q)$. Let us say that a polynomial $f(q) = a_0 + a_1 q + \cdots + a_d q^d$ is *palindromic* if $f(q) = q^d f(q^{-1})$, i.e., $a_i = a_{d-i}$ for $i = 0, \dots, d$.

Theorem 2. *(See Carrell–Peterson [4], also [2, Section 6.2].) For a permutation $w \in S_n$, the Schubert variety X_w is smooth if and only if the Poincaré polynomial $P_w(q)$ is palindromic.*

3. Inversion hyperplane arrangements

For a graph G on the vertex set $\{1, \ldots, n\}$, the *graphical arrangement* $\mathcal{A}_G$ is the hyperplane arrangement in $\mathbb{R}^n$ with hyperplanes $x_i - x_j = 0$ for all edges (i, j) in G. The *characteristic polynomial* $\chi_G(t)$ of the graphical arrangement $\mathcal{A}_G$ is also the *chromatic polynomial* of the graph G. The value of $\chi_G(t)$ at a positive integer t equals the number of *proper colorings* of the vertices of the graph G in t colors, i.e., the colorings such that all neighboring pairs of vertices have different colors. The value $(-1)^n \chi_G(-1)$ is the number of regions of $\mathcal{A}_G$. The regions of $\mathcal{A}_G$ are in bijection with acyclic orientations of the graph G. Recall that an *acyclic orientation* is a way to direct edges of G so that no directed cycles are formed. The region of $\mathcal{A}_G$ associated with an acyclic orientation $\mathcal{O}$ is described by the inequalities $x_i < x_j$ for all directed edges $i \to j$ in $\mathcal{O}$.

We will study a special class of graphical arrangements. For a permutation $w \in S_n$, the *inversion arrangement* $\mathcal{A}_w$ is the arrangement with hyperplanes $x_i - x_j = 0$ for each inversion $1 \leqslant i < j \leqslant n$, $w(i) > w(j)$. Define the *inversion graph* G_w as the graph on the vertex set $\{1, \ldots, n\}$ with the set of edges $\{(i, j) \mid i < j,\ w(i) > w(j)\}$. The arrangement $\mathcal{A}_w$ is the graphical arrangement $\mathcal{A}_G$ for the *inversion graph* $G = G_w$. Let R_w be the number of regions in the inversion arrangement $\mathcal{A}_w$.

Let $B_w := \#[\mathrm{id}, w] = P_w(1)$ be the number of elements in the Bruhat interval $[\mathrm{id}, w]$. Interestingly, the numbers R_w and B_w are related to each other.

Theorem 3. *(See Hultman, Linusson, Shareshian and Sjöstrand [8].)*

(1) *For any permutation $w \in S_n$, we have $R_w \leqslant B_w$.*
(2) *The equality $R_w = B_w$ holds if and only if w avoids the following four patterns* 4231*,* 35142*,* 42513*,* 351624*.*

This result was conjectured in [11] and announced as an open problem in a workshop in Oberwolfach in January 2007. A. Hultman, S. Linusson, J. Shareshian, and J. Sjöstrand proved the conjecture after the workshop.

Remark 4. It was shown in [11] that $R_w = B_w$ for all Grassmannian permutations w, which agrees with the above result. In this case, B_w counts the number of totally non-negative cells in the corresponding Schubert variety in the Grassmannian, see [11].

Remark 5. The four patterns from Theorem 3 came up earlier in the literature in at least two places. Firstly, Gasharov and Reiner [7] showed that the Schubert variety X_w can be described by simple inclusion conditions exactly when w avoids these four patterns. Secondly, Sjöstrand [12] showed that the Bruhat interval $[\mathrm{id}, w]$ can be described as the set of permutations associated with rook placements that fit inside a skew Ferrers board if and only if w avoids the same four patterns.

Remark 6. Note that each of the four patterns from Theorem 3 contains one of the two patterns from Lakshmibai–Sandhya's smoothness criterion. Thus the theorem implies the equality $R_w = B_w$ for all smooth permutations w.

4. Main results

Let us define the q-analog of the number of regions of the graphical arrangement $\mathcal{A}_G$, where G is a graph on the vertex set $\{1,\dots,n\}$. For two regions r and r' of the arrangement $\mathcal{A}_G$, let $d(r,r')$ be the number of hyperplanes in $\mathcal{A}_G$ that separate r and r'. In other words, $d(r,r')$ is the minimal number of hyperplanes we need to cross to go from r to r'. Let r_0 be the region of $\mathcal{A}_G$ that contains the point $(1,\dots,n)$. Define

$$R_G(q) := \sum_r q^{d(r,r_0)},$$

where the sum is over all regions r of the arrangement $\mathcal{A}_G$. Equivalently, the polynomial $R_G(q)$ can be described in terms of acyclic orientations of the graph G. For an acyclic orientation $\mathcal{O}$, let $\mathrm{des}(\mathcal{O})$ be the number of edges of G oriented as $i \to j$ in $\mathcal{O}$ where $i > j$ (descent edges). Then

$$R_G(q) = \sum_{\mathcal{O}} q^{\mathrm{des}(\mathcal{O})},$$

where the sum is over all acyclic orientations $\mathcal{O}$ of G. Indeed, for the acyclic orientation $\mathcal{O}$ associated with a region r we have $\mathrm{des}(\mathcal{O}) = d(r,r_0)$.

For $w \in S_n$, let $R_w(q) := R_{G_w}(q)$ be the polynomial that counts the regions of the inversion arrangement $\mathcal{A}_w = A_{G_w}$.

We are now ready to formulate the first main result of this paper. Recall that $P_w(q) := \sum_{u \leqslant w} q^{\ell(u)}$ is the Poincaré polynomial of the Schubert variety.

Theorem 7. *For a permutation $w \in S_n$, we have $P_w(q) = R_w(q)$ if and only if w is a smooth permutation, i.e., if and only if w avoids the patterns* 3412 *and* 4231.

This result was initially conjectured during a conversation of the second author (A.P.) with Vic Reiner.

The "only if" part of Theorem 7 is straightforward. Indeed, if w is not smooth, then by Carrell–Peterson's smoothness criterion (Theorem 2) the Poincaré polynomial $P_w(q)$ is not palindromic. On the other hand, the polynomial $R_w(q)$ is always palindromic, which follows from the involution on the regions induced by the map $x \mapsto -x$. Thus $P_w(q) \neq R_w(q)$ in this case. We will prove the "if" part of Theorem 7 in Section 6.

Our second result is an explicit non-recursive formula for the polynomials $P_w(q) = R_w(q)$, when w is smooth.

Let us say that an index $r \in \{1,\dots,n\}$ is a *record position* of a permutation $w \in S_n$ if $w(r) > \max(w(1),\dots,w(r-1))$. The values $w(r)$ are called the *records* or *left-to-right maxima* of w. For $i = 1,\dots,n$, let r and r' be the record positions of w such that $r \leqslant i < r'$ and there are no other record positions between r and r'. (Set $r' = +\infty$ if there are no record positions greater than i.) Let

$$e_i := \#\{j \mid r \leqslant j < i,\ w(j) > w(i)\} + \#\{k \mid r' \leqslant k \leqslant n,\ w(k) < w(i)\}.$$

Theorem 8. *Let w be a smooth permutation in S_n, and let $e_1,\dots,e_n$ be the numbers constructed from w as above. Then*

$$P_w(q) = R_w(q) = [e_1+1]_q[e_2+1]_q\cdots[e_n+1]_q.$$

Here $[a]_q := (1 - q^a)/(1 - q) = 1 + q + q^2 + \cdots + q^{a-1}$. We will prove Theorem 8 in Section 7.

Example 9. Let $w = 5164732$. The record positions of w are $1, 3, 5$. We have

$$(e_1, \ldots, e_7) = (0 + 3, 1 + 0, 0 + 2, 1 + 2, 0 + 0, 1 + 0, 2 + 0).$$

Theorem 8 says that $P_w(q) = R_w(q) = [4]_q[2]_q[3]_q[4]_q[1]_q[2]_q[3]_q$.

Remark 10. It was known before that the Poincaré polynomial $P_w(q)$ for smooth w factors as a product of q-numbers $[a]_q$. Gasharov [6] (see Proposition 21 below) gave a recursive construction for such factorization. On the other hand, Carrell gave a closed non-recursive expression for $P_w(q)$ as a ratio of two polynomials, see [4] and [2, Theorem 11.1.1]. However, it is not immediately clear from that expression that its denominator divides the numerator. One benefit of the formula in Theorem 8 is that it is non-recursive and it involves no division. Another combinatorial formula for $P_w(q)$ that has these features was given by Billey, see [1] and [2, Theorem 11.1.8].

5. Chordal graphs and perfect elimination orderings

A graph is called *chordal* if each of its cycles with four or more vertices has a *chord,* which is an edge joining two vertices that are not adjacent in the cycle. A *perfect elimination ordering* in a graph G is an ordering of the vertices of G such that, for each vertex v of G, all the neighbors of v that precede v in the ordering form a clique (i.e., a complete subgraph).

Theorem 11. *(See Fulkerson and Gross [5].) A graph is chordal if and only if it has a perfect elimination ordering.*

It is easy to calculate the chromatic polynomial $\chi_G(t)$ of a chordal graph G. Let us pick a perfect elimination ordering $v_1, \ldots, v_n$ of the vertices of G. For $i = 1, \ldots, n$, let e_i be the number of the neighbors of the vertex v_i among the preceding vertices $v_1, \ldots, v_{i-1}$. The numbers $e_1, \ldots, e_n$ are called the *exponents* of G. The following formula is well known.

Proposition 12. *The chromatic polynomial of the chordal graph G equals $\chi_G(t) = (t - e_1) \times (t - e_2) \cdots (t - e_n)$. Thus the graphical arrangement $\mathcal{A}_G$ has $(-1)^n \chi_G(-1) = (e_1 + 1) \times (e_2 + 1) \cdots (e_n + 1)$ regions.*

For completeness sake, we include the proof, which is also well known.

Proof. It is enough to prove the formula for a positive integer t. Let us count the number of proper coloring of vertices of G in t colors. The vertex v_1 can be colored in $t = t - e_1$ colors. Then the vertex v_2 can be colored in $t - e_2$ colors, and so on. The vertex v_i can be colored in $t - e_i$ colors, because the a_i preceding neighbors of v_i already used a_i *different* colors. □

Remark 13. A chordal graph can have many different perfect elimination orderings that lead to different sequences of exponents. However, the multiset (unordered sequence) $\{e_1, \ldots, e_n\}$ of the exponents does not depend on a choice of a perfect elimination order. Indeed, by Proposition 12, the exponents e_i are the roots of the chromatic polynomial $\chi_G(t)$.

Lemma 14. *(Cf. Björner, Edelman and Ziegler [3].) Suppose that a graph G on the vertex set $\{1, \dots, n\}$ has a vertex v adjacent to m vertices that satisfy the two conditions*:

(1) *The set of all neighbors of v is a clique in G.*
(2) (a) *All neighbors of v are less than v, or*
    (b) *all neighbors of v are greater than v.*

Then $R_G(q) = [m+1]_q R_{G\setminus v}(q)$, where $G \setminus v$ is the graph G with the vertex v removed.

This claim follows from general results of [3] on supersolvable hyperplanes arrangements. For completeness, we give a simple proof.

Proof. The polynomials $R_G(q)$ and $R_{G\setminus v}(q)$ are des-generating functions for acyclic orientations of the graphs G and $G \setminus v$.

Let us fix an acyclic orientation $\mathcal{O}$ of the graph $G \setminus v$, and count all ways to extend $\mathcal{O}$ to an acyclic orientation of G. The vertex v is connected to a subset S of m vertices of the graph $G \setminus v$, which forms the clique $G|_S \simeq K_m$. Clearly, there are $m+1$ ways to extend an acyclic orientation of the complete graph K_m to an acyclic orientation of K_{m+1}. Moreover, for each $j = 0, \dots, m$, there is a unique extension of $\mathcal{O}$ to an acyclic orientation $\mathcal{O}'$ of G such that there are exactly j edges oriented towards the vertex v in $\mathcal{O}'$ (and $m - j$ edges oriented away from v).

All vertices in S are less than v or all of them are greater than v. In both cases we have $\sum_{\mathcal{O}'} q^{\operatorname{des}(\mathcal{O}')} = [m+1]_q q^{\operatorname{des}(\mathcal{O})}$, where the sum is over extensions $\mathcal{O}'$ of $\mathcal{O}$. Thus $R_G(q) = [m+1]_q R_{G\setminus v}(q)$. $\square$

Definition 15. For a chordal graph G on the vertex set $\{1, \dots, n\}$, we say that a perfect elimination ordering $v_1, \dots, v_n$ of the vertices of G is *nice* if it satisfies the following additional property. For $i = 1, \dots, n$, all neighbors of the vertex v_i among the vertices $v_1, \dots, v_{i-1}$ are greater than v_i (in the usual order on $\mathbb{Z}$), or all neighbors of v_i among $v_1, \dots, v_{i-1}$ are less than v_i.

For a nice perfect elimination ordering $v_1, \dots, v_n$ of G, the last vertex $v = v_n$ satisfies the conditions of Lemma 14. Moreover, $v_1, \dots, v_{n-1}$ is a nice perfect elimination ordering of the graph $G \setminus v_n$. In this case, we can inductively use Lemma 14 to completely factor the polynomial $R_G(q)$ as $R_G(q) = [m+1]_q [m'+1]_q \cdots$. The numbers $m, m', \dots$ are exactly the exponents $e_n, e_{n-1}, \dots$ (written backwards) coming from this perfect elimination ordering.

Corollary 16. *Suppose that G has a nice perfect elimination ordering of vertices. Let $e_1, \dots, e_n$ be the exponents of G. Then we have*

$$R_G(q) = [e_1+1]_q [e_2+1]_q \cdots [e_n+1]_q.$$

6. Recurrence for polynomials $R_w(q)$

It is convenient to represent a permutation $w \in S_n$ as the *rook diagram* D_w, which is the placement of n non-attacking rooks into the boxes $(w(1), 1), (w(2), 2), \dots, (w(n), n)$ of the $n \times n$ board. See an example on Fig. 1. We assume that boxes of the board are labelled by pairs (i, j) in the same way as matrix elements. The rooks are marked by $\times$'s.

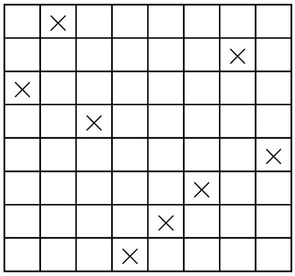


Fig. 1. The rook diagram D_w of the permutation $w = 31487625$.

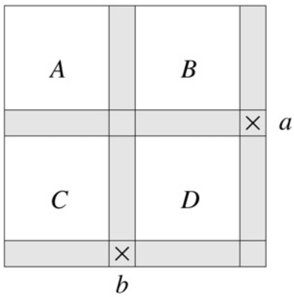
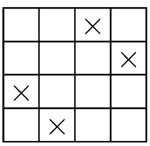
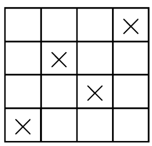


Fig. 2.

The inversion graph G_w contains an edge (i, j), with $i < j$, whenever the rook in the ith column of D_w is located to the South–West of the rook in the jth column. In this case, we say that this pair of rooks *forms an inversion.*

Here are the rook diagrams of the two forbidden patterns 3412 and 4231 for smooth permutations:





A permutation w is smooth if and only if its diagram D_w does not contain four rooks located in the same relative order as in one of these diagrams D_{3412} or D_{4231}.

Let a be the rook located in the last column of D_w, and let b be the rook located in the last row of D_w. The row containing a and the column containing b subdivide the diagram D_w into the four sectors A, B, C, D, as shown on Fig. 2. In the case when $w(n) = n$, we assume that $a = b$ and the sectors B, C, D are empty.

Lemma 17. *Let w be a smooth permutations. Then its rook diagram D_w has the following two properties*:

(1) *Each pair of rooks located in the sector D forms an inversion.*
(2) *At least one of the sectors B or C contains no rooks.*

For example, for the rook diagram $D_{31487625}$ shown on Fig. 1, the sector B contains one rook, the sector C contains no rooks, and the sector D contains two rooks that form an inversion.

Proof. (1) If the sector D contains a pair of rooks that do not form an inversion, then these two rooks together with the rooks a and b form a forbidden pattern as in the diagram D_{4231}.

(2) If the sector B contains at least one rook and the sector C contains at least one rook, then these two rooks together with the rooks a and b form a forbidden pattern as in the diagram D_{3412}. □

Let $v_a = n$ and v_b be the vertices of the inversion graph G_w corresponding to the rooks a and b. Also let $v_1, \dots, v_k$ be the vertices of G_w corresponding to the rooks inside the sector D.

If the sector B of the rook diagram D_w is empty, then the vertex v_b is connected only with the vertices $v_1, \dots, v_k, v_a$, that form a clique in the graph G_w, and all these vertices are greater than v_b. On the other hand, if the sector C of the rook diagram D_w is empty, then the vertex v_a is connected only with the vertices $v_b, v_1, \dots, v_k$, that form a clique, and all these vertices are less than v_a.

In both cases, the inversion graph G_w satisfies the conditions of Lemma 14, where $v = v_b$ if B is empty, and $v = v_a$ if C is empty. (If both B and C are empty, then we can pick $v = v_a$ or $v = v_b$.)

For $w \in S_n$ and $k \in \{1, \dots, n\}$, let $w' = \mathrm{flat}(w, k) \in S_{n-1}$ be the *flattening* of the sequence $w(1), \dots, w(k-1), w(k+1), \dots, w(n)$, that is, the permutation w' has the same relative order of elements as in this sequence. Equivalently, the rook diagram $D_{w'}$ is obtained from the rook diagram D_w by removing its kth column and the $w(k)$th row.

Lemma 14, together with the above discussion, implies the following recurrence relations for the polynomials $R_w(q)$.

Proposition 18. *Let $w \in S_n$ be a smooth permutation, and assume that $w(d) = n$ and $w(n) = e$. Then (at least) one of the following two statements is true:*

(1) $w(d) > w(d+1) > \cdots > w(n)$, *or*
(2) $w^{-1}(e) > w^{-1}(e+1) > \cdots > w^{-1}(n)$.

In both cases, the polynomial $R_w(q)$ factors as

$$R_w(q) = [m+1]_q R_{w'}(q),$$

where $w' = \mathrm{flat}(w, d)$ and $m = n - d$ in case (1), *or $w' = \mathrm{flat}(w, n)$ and $m = n - e$ in case* (2).

In this proposition, case (1) means that the sector B of the rook diagram D_w is empty, and case (2) means that the sector C is empty.

Clearly, if w is smooth, then the flattening $w' = \mathrm{flat}(w, k)$ is smooth as well. The inversion graph $G_{w'}$ is isomorphic to the graph $G \setminus k$. This means that, for smooth $w \in S_n$, one can inductively use Proposition 18 to completely factor the polynomial $R_w(q)$ as in Corollary 16.

Corollary 19. *For a smooth permutation $w \in S_n$, the inversion graph G_w is chordal and, moreover, it has a nice perfect elimination ordering. We have $R_w(q) = [e_1+1]_q[e_2+1]_q \cdots [e_n+1]_q$, where $e_1, \dots, e_n$ are the exponents of the inversion graph G_w.*

On the other hand, for any permutation $w \in S_n$, if the inversion graph G_w has a nice perfect elimination ordering, then w is smooth.

To prove the last claim note that if G_w has a nice perfect elimination ordering, then any induced subgraph of G_w has a nice perfect elimination ordering, so G_w cannot contain G_{3412} and G_{4231} as induced subgraphs, so w is smooth.

Remark 20. It is not true that G_w is chordal exactly when w is smooth. For example, for the non-smooth permutation $w = 4231$, the graph G_{4231} is chordal.

Interestingly, Gasharov [6] found exactly the same recurrence relations for the Poincaré polynomials $P_w(q)$.

Proposition 21. *(See Gasharov [6], cf. Lascoux [9].) The Poincaré polynomials $P_w(q)$, for smooth permutations w, satisfy exactly the same recurrence relation as in Proposition* 18.

Note that Lascoux [9] gave a factorization of the Kazhdan–Lusztig basis elements, that implies Proposition 21.

Propositions 18 and 21, together with the trivial claim $P_{\mathrm{id}}(q) = R_{\mathrm{id}}(q) = 1$, imply that $P_w(q) = R_w(q)$ for all smooth permutations w. This finishes the proof of Theorem 7.

7. Simple perfect elimination ordering

Section 6 gives a recursive construction for a nice perfect elimination ordering of the graph G_w, for smooth w. In this section we give a simple non-recursive construction for another perfect elimination ordering of G_w. This simple ordering may not be nice (see Definition 15). However, one still can use it for calculating the exponents of the graph G_w and factorizing the polynomials $P_w(q) = R_w(q)$ as in Corollary 19. Indeed, the multiset of the exponents does not depend on a choice of a perfect elimination ordering (see Remark 13).

Recall that a *record position* of a permutation $w \in S_n$ is an index $r \in \{1, \dots, n\}$ such that $w(r) > \max(w(1), \dots, w(r-1))$. Let $[a, b]$ denote the interval $\{a, a+1, \dots, b\}$ with the usual $\mathbb{Z}$-order of entries.

Lemma 22. *For a smooth permutation $w \in S_n$ with record positions $r_1 = 1 < r_2 < \cdots < r_s$, the ordering*

$$[r_s, n], [r_{s-1}, r_s - 1], \dots, [r_2, r_3 - 1], [r_1, r_2 - 1]$$

of the set $\{1, \dots, n\}$ is a perfect elimination ordering of the inversion graph G_w.

Example 23. (Cf. Example 9.) The permutation $w = 5164732$ has records 5, 6, 7 and record positions 1, 3, 5. Lemma 22 says that the ordering $\underline{5, 6, 7}$, $\underline{3, 4}$, $\underline{1, 2}$ is a perfect elimination ordering of the inversion graph G_w. Fig. 3 displays this inversion graph G_w. For each vertex $i = 1, \dots, 7$ of G_w, we wrote i inside a circle and $w(i)$ below it. The exponents of this graph (i.e., the numbers of edges going to the left from the vertices) are 0, 1, 2, 2, 3, 3, 1.

Proof of Lemma 22. Suppose that this ordering of vertices of G_w is not a perfect elimination ordering. This means that there is a vertex i connected in G_w with vertices j and k, preceding i in the order, such that the vertices j and k are not connected by an edge in G_w. Let us consider three cases.

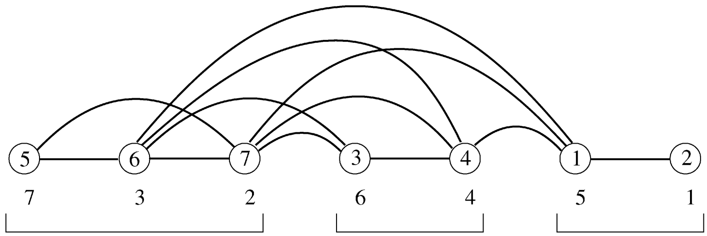


Fig. 3.

I. The vertices i, j, k belong to the same interval $I_p := [r_p, r_{p+1} - 1]$, for some $p \in \{1, \ldots, s\}$. (Here we assume that $r_{s+1} = n + 1$.) We have $k < j < i$ and $w(k) > w(i)$, $w(j) > w(i)$, but $w(k) < w(j)$, because (k, i) and (j, k) are edges of G_w but (k, j) is not an edge. The value $w(r_p)$ is the maximal value of w on the interval I_p. Since $w(k) < w(j)$ is not the maximal value of w on I_p, we have $r_p \neq k$ and so $r_p < k$. Thus $r_p < k < j < i$ and the values $w(r_p), w(k), w(j), w(i)$ form a forbidden 4231 pattern in w. So w is not smooth. Contradiction.

II. The vertices i, j are in the same interval I_p and the vertex k belongs to a different interval I_q. Then $q > p$, because the vertex k precedes i in the order. In this case we have $j < i < k$, $w(j) > w(i)$, $w(i) > w(k)$. This implies that $w(j) > w(k)$ that is (j, k) is an edge in the inversion graph G_w. Contradiction.

III. The vertex i belongs to the interval I_p and the vertices j, k do not belong to I_p. Assume that $j < k$ and that j belongs to I_q. Then $q > p$. In this case, $i < j < k$, $w(i) > w(j)$, $w(i) > w(k)$, and $w(j) < w(k)$. The record value $w(r_q)$ is greater than $w(i)$. This implies that $w(r_q) > w(i) > w(j)$. In particular, $w(r_q) \neq w(j)$ and, thus, $r_q \neq j$. We have $i < r_q < j < k$ and the values $w(i), w(r_q), w(j), w(k)$ form a forbidden 3412 pattern. Contradiction. $\square$

Proof of Theorem 8. Let us calculate the exponents of the inversion graph G_w for a smooth permutation $w \in S_n$ using the perfect elimination ordering from Lemma 22. Suppose that $i \in I_p$. Then the exponent e_i of the vertex i equals the number of neighbors of the vertex i in the graph G_w among the preceding vertices, that is among the vertices in the sets $\{r_p, \ldots, i-1\}$ and $I_{p+1} \cup I_{p+2} \cup \cdots$. In other words, the exponent e_i equals

$$\#\{j \mid r_p \leqslant j < i,\ w(j) > w(i)\} + \#\{k \mid k \geqslant r_{p+1},\ w(k) < w(i)\}.$$

This is exactly the expression for e_i from Theorem 8. The result follows from Corollary 19. $\square$

8. Final remarks

Our proof of Theorem 7 is based on a recurrence relation. It would be interesting to give more direct combinatorial proof of Theorem 7 based on a bijection between elements of the Bruhat interval [id, w] and regions of the arrangement $\mathcal{A}_w$.

It would be interesting to better understand the relationship between Bruhat intervals [id, w] and the hyperplane arrangement $\mathcal{A}_w$. One can construct a directed graph Γ_w on the regions of $\mathcal{A}_w$. Two regions r and r' are connected by a directed edge (r, r') if these two regions are adjacent (i.e., separated by a single hyperplane) and r is more close to r_0 than r'. For example, for the longest permutation w_0, the graph Γ_{w_0} is the Hasse diagram of the *weak* Bruhat order. Is it true that, for any smooth permutation $w \in S_n$, the graph Γ_w is isomorphic to a subgraph of the Hasse diagram of the Bruhat interval [id, w]?

It would be interesting to explain Theorem 7 from a geometrical point of view. Is it possible to link the arrangement $\mathcal{A}_w$ and the polynomial $R_w(q)$ with the cohomology ring of the Schubert variety X_w? Is it possible to define a related ring structure on the regions of $\mathcal{A}_w$?

The statement of Theorem 7 can extended to any finite Weyl group W, as follows. For a Weyl group element $w \in W$, let $P_w(q) := \sum_u q^{\ell(w)}$, where the sum is over all $u \in W$ such that $u \leqslant w$ in the Bruhat order on W. Define the arrangement $\mathcal{A}_w$ as the collection of hyperplanes $\alpha(x) = 0$ for all roots α in the corresponding root system such that $\alpha > 0$ and $w(\alpha) < 0$. Let r_0 be the region of $\mathcal{A}_w$ that contains the fundamental chamber of the corresponding Coxeter arrangement. Define $R_w(q) := \sum_r q^{d(r_0,r)}$, where the sum is over all regions of the arrangement $\mathcal{A}_w$ and $d(r_0, r)$ is the number of hyperplanes separating r_0 and r. Let $X_w = BwB/B$ be the Schubert variety in the corresponding generalized flag manifold G/B. Details about (rational) smoothness of Schubert varieties X_w can be found in [2].

Conjecture 24. *The equality $P_w(q) = R_w(q)$ holds if and only if the Schubert variety X_w is rationally smooth.*

Finally, let us mention that the inverse of Corollary 16 might be true.

Conjecture 25. *For a graph G on the vertex set $\{1, \ldots, n\}$, the polynomial $R_G(q)$ can be factorized as a product of q-numbers if and only if the graph G has a nice perfect elimination order.*

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