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Stable multivariate Eulerian polynomials and generalized Stirling permutations

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ABSTRACT

We study Eulerian polynomials as the generating polynomials of the descent statistic over Stirling permutations—a class of restricted multiset permutations. We develop their multivariate refinements by indexing variables by the values at the descent tops, rather than the position where they appear. We prove that the obtained multivariate polynomials are stable, in the sense that they do not vanish whenever all the variables lie in the open upper half-plane. Our multivariate construction generalizes the multivariate Eulerian polynomial for permutations, and extends naturally to r-Stirling and generalized Stirling permutations.

The benefit of this refinement is manifold. First of all, the stability of the multivariate generating functions implies that their univariate counterparts, obtained by diagonalization, have only real roots. Second, we obtain simpler recurrences of a general pattern, which allows for essentially a single proof of stability for all the cases, and further proofs of equidistributions among different statistics. Our approach provides a unifying framework of some recent results of Bóna, Brändén, Brenti, Janson, Kuba, and Panholzer. We conclude by posing several interesting open problems.

1. Introduction

The polynomials

“$\alpha = x$

$\beta = x + x^2$

$\gamma = x + 4x^2 + x^3$

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$$\delta = x + 11x^2 + 11x^3 + x^4$$
$$\varepsilon = x + 26x^2 + 66x^3 + 26x^4 + x^5$$
$$\zeta = x + 57x^2 + 302x^3 + 302x^4 + 57x^5 + x^6 \text{ etc.''}$$

appeared in Euler's work on a method of summation of series [16]. Since then these polynomials, known as the Eulerian polynomials and their coefficients, the so-called Eulerian numbers have been widely studied in enumerative combinatorics, especially within the combinatorics of permutations. They serve as generating functions of several statistics such as descents, exceedances, or runs of permutations. They are intimately connected to Stirling numbers, and also the binomial coefficients, via the famous Worpitzky-identity. See the notes of Foata and Schützenberger [17,18] for a survey and the history of these polynomials.

It is often useful to consider multivariate refinements of polynomials. In many instances with the help of such refinements more general results can be obtained with significantly shorter and simpler proofs. This is especially the case if the multivariate polynomials satisfy some additional property, for example, multiaffine and homogeneous polynomials are relatively easy to handle. See [4,22,37] for a few of the numerous recent successes using multivariate generalizations.

In this paper, we study statistics over Stirling permutations — a class of restricted multiset permutations, defined by Gessel and Stanley [20]. We give a multivariate refinement of the descent generating polynomial, by indexing variables based on the values at the descent tops, rather than the positions where they appear. Our construction generalizes the one in [23] for multivariate Eulerian polynomials and is further extended to joint statistic generating polynomials over r-Stirling permutations, and generalized Stirling permutations. We prove that these polynomials are stable strengthening recent results of Bóna [2], Brändén et al. [8], Brenti [9], and Janson et al. [25,26].

1.1. Organization

In Section 2, we give the necessary definitions and discuss previous work. We define the statistics over permutations and Stirling permutations that we study, and the related Eulerian numbers and polynomials. We give theorems that state that the roots of the Eulerian polynomials and the second-order Eulerian polynomials are all real. We define the notion of stability, a multivariate generalization of real-rootedness, and review some results from the theory of stable polynomials. We give Brändén's proof of a closely related multivariate generating polynomial.

In Section 3, we begin with a multivariate refinement of the Eulerian polynomial that simultaneously refines both the descent and the weak exceedance statistic. Next we apply Brändén's proof to show stability for the multivariate Eulerian polynomial. We then generalize this result to Stirling permutations, by proving that the multivariate refinement of the second-order Eulerian polynomial is also stable. In fact, the recursion satisfied by these multivariate polynomials can be modeled by a (generalized) Pólya urn, one considered by Janson et al. in [26]. This model allows us to further extend our results to r-Stirling permutations and even generalized Stirling permutations. By appropriately defining new statistics for these objects, we obtain more general multivariate Eulerian polynomials and stability results that contain the previous ones as special cases. As a corollary, we also obtain a multivariate generalization of a theorem of Brenti on the real-rootedness of descent generating polynomials over generalized Stirling permutations.

Finally, in Section 4, we discuss further connections to Legendre–Stirling permutations, q-analogs, Durfee squares and present some open questions.

2. Previous work

2.1. Statistics on permutations and Stirling permutations

Let n be a positive integer, and let $\mathfrak{S}_n$ denote the set of all permutations of the set $\{1, \ldots, n\}$. For a permutation $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$, let

$$\mathcal{A}(\pi) = \{i \mid \pi_{i-1} < \pi_i\},$$
$$\mathcal{D}(\pi) = \{i \mid \pi_i > \pi_{i+1}\},$$

with $\pi_0 = \pi_{n+1} = 0$, denote the ascent set and the descent set of π, respectively. One way to think about these sets is to consider the permutation π padded with zeros from the front and the back. Note that with this convention, $i = 1$ is always an ascent and $i = n$ is always a descent. Also note that i is the index of the larger element of the two, both in the definition of the ascent set and the descent set. We will call π_i with $i \in \mathcal{D}(\pi)$ a descent top, and similarly π_j with $j \in \mathcal{A}(\pi)$ an ascent top. We use $\mathrm{des}(\pi) = |\mathcal{D}(\pi)|$ and $\mathrm{asc}(\pi) = |\mathcal{A}(\pi)|$ to denote the cardinality of these sets, the number of descents and ascents in π, respectively.

Gessel and Stanley defined the following restricted subset of multiset permutations called *Stirling permutations* in [20]. Let $\mathcal{Q}_n$ denote the set of permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ in which, for all i, all entries between the two occurrences of i are larger than i. For instance, $\mathcal{Q}_1 = \{11\}$, $\mathcal{Q}_2 = \{1122, 1221, 2211\}$, and from a recursive construction rule – observe that n and n have to be adjacent in $\mathcal{Q}_n$ – it is not difficult to see that $|\mathcal{Q}_n| = 1 \cdot 3 \cdot \cdots \cdot (2n-1) = (2n-1)!!$. Gessel and Stanley also studied the descent statistic over $\mathcal{Q}_n$. The notions of ascents and descents can be easily extended to Stirling permutations. Bóna in [2] introduced an additional statistic called *plateau* and studied the distribution of the following three statistics over Stirling permutations. For $\sigma = \sigma_1\sigma_2 \ldots \sigma_{2n} \in \mathcal{Q}_n$, let

$$\mathcal{A}(\sigma) = \{i \mid \sigma_{i-1} < \sigma_i\},$$
$$\mathcal{D}(\sigma) = \{i \mid \sigma_i > \sigma_{i+1}\},$$
$$\mathcal{P}(\sigma) = \{i \mid \sigma_i = \sigma_{i+1}\}$$

denote the set of ascents, descents and plateaux of σ, respectively. As before, we pad the Stirling permutation with zeros, i.e., we define $\sigma_0 = \sigma_{2n+1} = 0$. So, for $\sigma \in \mathcal{Q}_n$, $i = 1$ is always an ascent and $i = 2n$ is always a descent. We will use $\mathrm{asc}(\sigma) = |\mathcal{A}(\sigma)|$, $\mathrm{des}(\sigma) = |\mathcal{D}(\sigma)|$ and $\mathrm{plat}(\sigma) = |\mathcal{P}(\sigma)|$ to denote the number of ascents, descents, and plateaux in σ.

Note that, the sum of the number of ascents, descents and plateaux in any $\sigma \in \mathcal{Q}_n$ is $2n + 1$, the number of gaps (counting the padding zeros). Interestingly, the three statistics are equidistributed over $\mathcal{Q}_n$, as was shown in [2].

2.2. Eulerian numbers and polynomials

Eulerian numbers (see sequence A008292 in the OEIS [36]) denoted by $\left\langle {n \atop k} \right\rangle$, or sometimes by $A(n, k)$, are amongst the most studied sequences of numbers in enumerative combinatorics. They count, for example, the number of permutations of $\{1, \ldots, n\}$ with k descents:

$$\left\langle {n \atop k} \right\rangle := |\{\pi \in \mathfrak{S}_n \mid \mathrm{des}(\pi) = k\}|. \tag{1}$$

Note that here $1 \le k \le n$, since $\pi_0 = \pi_{n+1} = 0$ in our definition of descent. This indexing of the Eulerian numbers is in correspondence with the classic books by Comtet [13], Riordan [31], and Stanley [38].

It can be deduced from the definition in (1), that the Eulerian numbers satisfy the following recursion:

$$\left\langle {n+1 \atop k} \right\rangle = k \left\langle {n \atop k} \right\rangle + (n + 2 - k) \left\langle {n \atop k-1} \right\rangle, \tag{2}$$

for $2 \le k \le n+1$, with initial condition $\left\langle {1 \atop 1} \right\rangle = 1$ and boundary conditions $\left\langle {n \atop k} \right\rangle = 0$ for $k \le 0$ or $n < k$. In this paper, we investigate their ordinary generating function, the *Eulerian polynomials*:

$$A_n(x) = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle x^k = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)}, \tag{3}$$

along with several generalizations of them.

A consequence of the recursion for the Eulerian numbers (2) is a recursion for the Eulerian polynomials, namely, for any $n \geq 1$,

$$A_{n+1}(x) = (n+1)xA_n(x) + x(1-x)A_n'(x). \tag{4}$$

This recursion gives rise to the following classical result already noted by Frobenius [19].

Theorem 2.1. *$A_n(x)$ has only real roots. (In addition, the roots are all distinct, and nonpositive.)*

See [3, Theorem 1.33] for a proof using Rolle's theorem.

2.3. Second-order Eulerian numbers

We adopt the notation

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle := |\{\sigma \in \mathcal{Q}_n \mid \mathrm{des}(\sigma) = k\}| \tag{5}$$

with $1 \leq k \leq n$. Following [21], we refer to these numbers as the "second-order Eulerian numbers"[1] since they satisfy a recursion (see [20], for example) very similar to (2). For $2 \leq k \leq n+1$, we have

$$\left\langle\!\!\left\langle {n+1 \atop k} \right\rangle\!\!\right\rangle = k\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle + (2n+2-k)\left\langle\!\!\left\langle {n \atop k-1} \right\rangle\!\!\right\rangle, \tag{6}$$

with initial condition $\left\langle\!\left\langle {1 \atop 1} \right\rangle\!\right\rangle = 1$ and boundary conditions $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle = 0$ for $k \leq 0$ or $n < k$. Note that our indexing is in agreement with sequence A008517 in [36] and with the definition of the statistics adopted from [2], however it differs from the one in [21].

The second-order Eulerian numbers are not as well studied as the Eulerian numbers. Nevertheless, they are known to have several interesting combinatorial interpretations. Apart from counting Stirling permutations $\mathcal{Q}_n$ with k descents [20], k ascents, k plateaux [2], these numbers also count the number of Riordan trapezoidal words of length n with k distinct letters [32], the number of rooted plane trees on $n+1$ nodes with k leaves [25], and matchings of the complete graph on $2n$ vertices with $n-k$ left-nestings (Claim 6.1 of [27]).

Bóna proved the following theorem (analogous to Theorem 2.1) on the roots of the ordinary generating function of the second-order Eulerian numbers.

Theorem 2.2 (*Theorem 1 of [2]*)**.** *$C_n(x) = \sum_{k=1}^{n} \left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle x^k$ has only real (simple, nonnegative) roots.*

Observe that the following recursion (given in [32])

$$C_{n+1}(x) = (2n+1)xC_n(x) + x(1-x)C_n'(x) \tag{7}$$

satisfied by these generating polynomials is strikingly similar to (4).

2.4. Polynomials with only real roots

For a generating polynomial to have only real roots is an important property in combinatorics. It is often used to show that a (nonnegative) sequence $\{b_i\}_{i=0\ldots n}$ is log-concave ($b_i^2 \geq b_{i+1}b_{i-1}$) and unimodal ($b_0 \leq \cdots \leq b_k \geq \cdots \geq b_n$). If the generating polynomial $\sum_{i=0}^{n} b_i x^i$ has only real roots, then the above properties hold for the coefficients. In fact, even more is true, e.g., there are at most two modes, and the coefficients satisfy several nice properties such as Newton's inequalities, Darroch's theorem, etc. Furthermore, the normalized coefficients $b_i/(\sum_i b_i)$ – viewed as a probability distribution – converge to a normal distribution as n goes to infinity, under the additional constraint that the variance tends to infinity [1,24].

[1] Not to be confused with $2^n - 2n$ (see sequence A005803 in [36]).

In what follows, we give a generalization of the Eulerian polynomials $A_n(x)$ to multiple variables, in such a way that a multivariate analog of Theorem 2.1 holds for them (in fact, the multivariate theorem will contain the univariate version as a special case). For this, we need a generalization of the real-rooted property for polynomials in multiple variables, the notion of stability with which we continue next.

2.5. Stable polynomials and stability preservers

We call a polynomial $f(z_1, \ldots, z_m) \in \mathbb{R}[z_1, \ldots, z_m]$ *stable*, if whenever $\Im(z_i) > 0$ for all i then f does not vanish. Note that a univariate polynomial $f(z) \in \mathbb{R}[z]$ has only real roots if and only if it is stable.

Thanks to the recent work of Borcea and Brändén, the theory of stable polynomials has evolved into a very applicable technique. For a concise collection of the latest results and some recent applications of the theory, we refer to the survey of Wagner [39] and the references therein.

In this paper, we rely on Borcea and Brändén's characterization of linear operators that preserve stability. Such operators map stable polynomials to stable ones or to the identically 0 polynomial. Our proofs are based on the following two results on stability preserving operators.

Lemma 2.3 (*Cf. Lemma 2.4 of [39]*). *The following operations preserve stability of polynomials in* $\mathbb{R}[z_1, \ldots, z_n]$.

(a) *Permutation: for any permutation* $\sigma \in \mathfrak{S}_n$, $f \mapsto f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$.
(b) *Diagonalization: for* $1 \le i < j \le n$, $f \mapsto f(z_1, \ldots, z_n)|_{z_i = z_j}$.
(c) *Specialization: for* $a \in \mathbb{R}$, $f \mapsto f(a, z_2, \ldots, z_n)$.
(d) *Translation:* $f \mapsto f(z_1 + t, z_2, \ldots, z_n) \in \mathbb{R}[z_1, \ldots, z_n, t]$.
(e) *Differentiation:* $f \mapsto \partial f / \partial z_1$.

We call a multivariate polynomial *multiaffine* if it has degree at most 1 in each variable. Theorem 2.2 of [5] gives a complete characterization of real stability preservers for multiaffine polynomials. We will only need one part of the theorem, the following sufficient condition.

Lemma 2.4 (*Cf. Theorem 2.2 of [5]*). *Let* $f \in \mathbb{C}[z_1, \ldots, z_n]$ *be a stable multiaffine polynomial and let* T *denote a linear operator acting on the* $z_1, \ldots, z_n$ *variables. Suppose that* $G_T = T(\prod_{i=1}^{n}(z_i + w_i)) \in \mathbb{C}[z_1, \ldots, z_n, w_1, \ldots, w_n]$ *is a stable polynomial. Then* $T(f)$ *is either stable or identically* 0.

2.6. Towards a stable multivariate Eulerian polynomial

Brändén and Stembridge suggested finding a stable multivariate generalization of the Eulerian polynomials. In [23], the polynomial

$$A_n(\mathbf{x}) = \sum_{\pi \in \mathfrak{S}_n} \prod_{\pi_i \ge i} x_{\pi_i}, \tag{8}$$

with $\mathbf{x} = x_1, \ldots, x_n$ was conjectured to be stable. It was also proven to be stable for $n \le 5$.

Brändén considered the closely related multivariate generating polynomial[2]:

$$\tilde{A}_n(\mathbf{x}) = \sum_{\pi \in \mathfrak{S}_n} \prod_{\pi_i > \pi_{i+1}} x_{\pi_i}, \tag{9}$$

and the following homogeneous extension of it:

$$\tilde{A}_n(\mathbf{x}, \mathbf{y}) = \sum_{\pi \in \mathfrak{S}_n} \prod_{\pi_i > \pi_{i+1}} x_{\pi_i} \prod_{\pi_i < \pi_{i+1}} y_{\pi_{i+1}}, \tag{10}$$

and proved the following [6].

[2] We use $\tilde{A}$ to emphasize that this generating polynomial has degree one less than the Eulerian polynomials defined in (8). In particular, $A_n(x, \ldots, x) = x\tilde{A}_n(x, \ldots, x)$, which is sometimes referred to in the literature as the *classical* Eulerian polynomial.

Theorem 2.5. *$\tilde{A}_n(\mathbf{x}, \mathbf{y})$ is a stable polynomial.*

For the sake of completeness, we give the proof.

Proof. The proof is by induction. The statement holds for the base case $n = 1$, since the polynomial $\tilde{A}_1(x_1, y_1) = 1$ is stable. For $n \geq 1$, the polynomials satisfy the following recurrence:

$$\tilde{A}_{n+1}(\mathbf{x}, \mathbf{y}) = (x_{n+1} + y_{n+1})\tilde{A}_n(\mathbf{x}, \mathbf{y}) + x_{n+1}y_{n+1}\partial\tilde{A}_n(\mathbf{x}, \mathbf{y}), \tag{11}$$

where $\partial = \sum_{i=1}^n \partial/\partial x_i + \sum_{j=1}^n \partial/\partial y_j$. This can be seen by observing the effect on the sets $\{\pi_i \mid \pi_{i-1} < \pi_i\}$ and $\{\pi_i \mid \pi_i > \pi_{i+1}\}$ of inserting $n + 1$ into a permutation $\pi \in \mathfrak{S}_n$. Now the statement follows from the fact that $T = (x_{n+1} + y_{n+1}) + x_{n+1}y_{n+1}\partial$ is a stability preserving operator (by an application of Lemma 2.4). □

By letting $y_i = 1$ for all i, and applying Lemma 2.3 we obtain the following.

Corollary. *$\tilde{A}_n(\mathbf{x})$ is a stable polynomial.*

The multivariate Monotone Column Permanent Theorem (Theorem 3.4 of [8]) contains Theorem 2.5 as the special case for a Ferrers matrix (equivalently, a Ferrers board) of staircase shape. As a consequence of this matrix interpretation, and a bijection of Riordan, it was also noted in [8] that

$$\tilde{A}_n(\mathbf{x}) = \sum_{\pi \in \mathfrak{S}_n} \prod_{\pi_i > i} x_{\pi_i}. \tag{12}$$

3. Our results

We first note that a similar result holds for the multivariate Eulerian polynomial defined in (8) as well. Namely, the multivariate refinement $A_n(\mathbf{x})$ proposed in [23] refines the descent and weak exceedance statistics simultaneously.

Proposition 3.1.

$$\begin{aligned} A_n(\mathbf{x}) &= \sum_{\pi \in \mathfrak{S}_n} \prod_{i \in \mathcal{D}(\pi)} x_{\pi_i} \\ &= \sum_{\pi \in \mathfrak{S}_n} \left(\prod_{\pi_i > \pi_{i+1}} x_{\pi_i} \right) x_{\pi_n}. \end{aligned}$$

Proof. Follows from a modification of the above mentioned bijection of Riordan [31]. □

The second line of the equation is given to highlight the difference between this formula and the one in (9), since $n \in \mathcal{D}(\pi)$ in our notation.

Consider the following homogenization of $A_n(\mathbf{x})$:

$$A_n(\mathbf{x}, \mathbf{y}) = \sum_{\pi \in \mathfrak{S}_n} \prod_{i \in \mathcal{D}(\pi)} x_{\pi_i} \prod_{i \in \mathcal{A}(\pi)} y_{\pi_i}. \tag{13}$$

Theorem 3.2. *$A_n(\mathbf{x}, \mathbf{y})$ is a stable polynomial.*

Proof. $A_1(x_1, y_1) = x_1y_1$ which is stable. Note that the following recursion

$$A_{n+1}(\mathbf{x}, \mathbf{y}) = x_{n+1}y_{n+1}\partial A_n(\mathbf{x}, \mathbf{y}), \tag{14}$$

holds for $n \geq 1$, where ∂ again denotes the sum of all partials. Since ∂ is a stability preserving operator, the same inductive proof as in Theorem 2.5 goes through. □

We note that indexing by the values π_i at the ascent tops and descent tops is crucial. If instead the descents and ascents were indexed by the position i where they appear, the polynomials would fail to be stable. From Theorem 3.2 we get the following corollary, which is also a special case of Theorem 3.4 of [8] for Ferrers boards of staircase shape.

Corollary. *$A_n(\mathbf{x})$ is a stable polynomial.*

In [8], the stability results for $\tilde{A}_n(\mathbf{x})$ were further extended to obtain a stable multivariate generalization of the multiset Eulerian polynomial previously studied by Simion [35]. We continue along a similar direction as well, and extend Theorem 3.2 to a restricted subset of multiset permutations, the Stirling permutations.

3.1. Multivariate second-order Eulerian polynomials

Janson in [25] suggested studying the trivariate polynomial that simultaneously counts all three statistics of the Stirling permutations (see also [14]):

$$C_n(x, y, z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} z^{\mathrm{plat}(\sigma)}. \tag{15}$$

We go a step further, and introduce a refinement of this polynomial obtained by indexing each ascent, descent and plateau by the value where they appear, i.e., ascent top, descent top, plateau:

$$C_n(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{\sigma \in \mathcal{Q}_n} \prod_{i \in \mathcal{D}(\sigma)} x_{\sigma_i} \prod_{i \in \mathcal{A}(\sigma)} y_{\sigma_i} \prod_{i \in \mathcal{P}(\sigma)} z_{\sigma_i}. \tag{16}$$

For example, $C_1(\mathbf{x}, \mathbf{y}, \mathbf{z}) = x_1 y_1 z_1$, $C_2(\mathbf{x}, \mathbf{y}, \mathbf{z}) = x_2 y_1 y_2 z_1 z_2 + x_1 x_2 y_1 y_2 z_2 + x_1 x_2 y_2 z_1 z_2$.

These polynomials are multiaffine, since any value $v \in \{1, \dots, n\}$ can only appear at most once as an ascent top (similarly, at most once as a descent top or a plateau, respectively). This is immediate from the restriction in the definition of a Stirling permutation. Furthermore, each gap $(j, j+1)$ for $0 \le j \le 2n$ in a Stirling permutation $\sigma \in \mathcal{Q}_n$ is either a descent, or an ascent or a plateau. This implies that $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is also homogeneous, and of degree $2n + 1$.

Theorem 3.3. *The polynomial $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$ defined in (16) is stable.*

Proof. Note that

$$C_{n+1}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = x_{n+1} y_{n+1} z_{n+1} \partial C_n(\mathbf{x}, \mathbf{y}, \mathbf{z}), \tag{17}$$

where $\partial = \sum_{i=1}^n \partial/\partial x_i + \sum_{i=1}^n \partial/\partial y_i + \sum_{i=1}^n \partial/\partial z_i$. The recursion follows from the fact that each Stirling permutation in $\mathcal{Q}_{n+1}$ is obtained by inserting $(n+1)(n+1)$ into one of the $2n+1$ gaps of some $\sigma \in \mathcal{Q}_n$. This insertion introduces a new ascent, a new plateau, a new descent, and removes the statistic – either ascent, plateau, or descent – that existed in the gap before. From here, the proof is analogous to that of Theorem 2.5. □

There are some interesting corollaries of this theorem. First, note that, diagonalizing variables preserves stability (see part b in Lemma 2.3). Hence, by setting $x_1 = \cdots = x_n = x$, $y_1 = \cdots = y_n = y$, and $z_1 = \cdots = z_n = z$, we immediately have the following.

Corollary. *The trivariate generating polynomial $C_n(x, y, z)$ defined in (15) is stable.*

Specializing variables also preserves stability (part c of Lemma 2.3). Thus, by setting $y = z = 1$, we get back Theorem 2.2.

If we specialize variables first, without diagonalizing, namely set $y_1 = \cdots = y_n = z_1 = \cdots = z_n = 1$ we get a different corollary.

Corollary. *The multivariate descent polynomial for Stirling permutations*

$$C_n(x_1, \dots, x_n) = \sum_{\sigma \in \mathcal{Q}_n} \prod_{i \in \mathcal{D}(\sigma)} x_{\sigma_i} \tag{18}$$

is stable.

From the symmetry of the recursion (17) and the fact that $C_1(x, y, z) = xyz$ we also get the following.

Corollary (*Theorem 2.1 of [25]*)**.** *The trivariate polynomial $C_n(x, y, z)$ defined in (15) is symmetric in the variables x, y, z.*

Corollary (*Proposition 1 of [2]*)**.** *The ascents, descents and plateaux are equidistributed over $\mathcal{Q}_n$.*

3.2. Generalized Pólya urns and generalized Stirling permutations

One can model the differential recursion in (17) as follows (see the Urn I model in [25]). Step 1: start with $r = 3$ balls in an urn. Each ball has a different color: red, green, blue. At each step i, for $i = 2 \dots n$, we remove one ball (chosen uniformly at random) from the urn and put in three new balls, one of each color. The distribution of the balls of each color corresponds to the distribution of the ascents (red), descents (green), and plateaux (blue) in a Stirling permutation. Our multivariate refinement can be thought of as simply labeling each ball by a number i that represents the step i when we placed the ball in the urn. Clearly, this method can be further generalized to r colors, as was done by Janson et al. in [26], which led them to consider statistics over generalizations of Stirling permutations.

We begin with one such generalization suggested by Gessel and Stanley [20], called *r-Stirling permutations*, which have also been studied by Park in [28–30] under the name r-multipermutations. An r-Stirling permutation of order n is a generalized Stirling permutation of the multiset $\{1^r, \dots, n^r\}$. Formally, let r be a positive integer, and let $\mathcal{Q}_n(r)$ denote the set of multiset permutations of $\{1^r, \dots, n^r\}$ with the property that all elements between two occurrences of i are at least i. In other words, every element that appears between "consecutive" occurrences of i is larger than i, or in pattern avoidance terminology, $\mathcal{Q}_n$ consists of multiset permutations of $\{1^r, \dots, n^r\}$ that are 212-avoiding.

Janson et al. in [26] considered various statistics over r-Stirling permutations. We define the ascents and descents identically as in the two previous sections (with the convention of padding with zeros, $\sigma_0 = \sigma_{rn+1} = 0$). In addition, we will adopt their definition of the j-plateau, which is as follows. For an r-Stirling permutation σ, a *j-plateau* of σ, denoted by $\mathcal{P}_j(\sigma)$, is the set of indices i such that $\sigma_i = \sigma_{i+1}$ where $\sigma_1 \dots \sigma_{i-1}$ contains $j-1$ instances of σ_i. In other words, a j-plateau counts the number of times the jth occurrence of an element is followed immediately by the $(j+1)$st occurrence of it. We note that there are j-plateaux for $j = 1 \dots r-1$ in $\mathcal{Q}_n(r)$.

Now we can define a multivariate polynomial that is analogous to the previously studied $A_n(\mathbf{x}, \mathbf{y})$ and $C_n(\mathbf{x}, \mathbf{y}, \mathbf{z})$. For $r \geq 1$, let

$$E_n(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots, \mathbf{z}_{r-1}) = \sum_{\sigma \in \mathcal{Q}_n(r)} \left(\prod_{i \in \mathcal{D}(\sigma)} x_{\sigma_i} \right) \left(\prod_{i \in \mathcal{A}(\sigma)} y_{\sigma_i} \right) \prod_{j=1}^{r-1} \left(\prod_{i \in \mathcal{P}_j(\sigma)} z_{j,\sigma_i} \right), \tag{19}$$

where $\mathbf{z}_j = z_{j,1}, \dots, z_{j,n}$ for all $j = 1, \dots, r-1$.

Theorem 3.4. *$E_n(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots, \mathbf{z}_{r-1})$ is a stable polynomial.*

The proof is identical to that of A_n and C_n (see Theorems 3.2 and 3.3). As a corollary, we obtain that the diagonalized polynomial, $E_n(x, y, z_1, \dots, z_{r-1})$ is symmetric in the variables $x, y, z_1, \dots, z_{r-1}$ which implies the results of Theorem 9 in [26]. Analogously, we could define the rth order Eulerian numbers as the number of r-Stirling permutations with exactly k descents (or equivalently, k ascents or k j-plateaux for some fixed j). We suggest the notation $\left\langle {n \atop k} \right\rangle_r$, r being the shorthand for the r angle parentheses. The results for the special cases of $r = 1$ and $r = 2$ give the results for permutations and Stirling permutations, respectively.

Janson et al. in [26] also studied statistics over generalized Stirling permutations. These permutations were previously investigated by Brenti in [9,10]. The set of *generalized Stirling permutations* of rank n, denoted by $\mathcal{Q}_n^*$, is the set of all permutations of the multiset $\{1^{k_1}, \dots, n^{k_n}\}$ with the same restriction as before. Namely, that for each i, for $1 \leq i \leq n$, the elements occurring between two occurrences of i are at least i.

We can further generalize the multivariate Eulerian polynomials by simply extending the above defined statistics to generalized Stirling permutations. This corresponds to an urn model with balls colored with $\kappa = \max_{i=1}^n k_i + 1$ many colors: $c_1, c_2, \dots, c_\kappa$. We start with $k_1 + 1$ balls in the urn colored with $c_1, \dots, c_{k_1+1}$ (each ball has a different color). In each round i, for $2 \leq i \leq n$, we remove one ball and put in $k_i + 1$ balls, one from each of the first $k_i + 1$ colors, $c_1, \dots, c_{k_i+1}$. We can then define a multivariate polynomial counting all statistics simultaneously,

$$E_n(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots, \mathbf{z}_{\kappa-2}) = \sum_{\sigma \in \mathcal{Q}_n^*} \left(\prod_{i \in \mathcal{D}(\sigma)} x_{\sigma_i} \right) \left(\prod_{i \in \mathcal{A}(\sigma)} y_{\sigma_i} \right) \prod_{j=1}^{\kappa-2} \left(\prod_{i \in \mathcal{P}_j(\sigma)} z_{j,\sigma_i} \right). \tag{20}$$

Theorem 3.5. $E_n(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots, \mathbf{z}_{\kappa-2})$ *is stable.*

Proof.

$$E_{n+1}(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots, \mathbf{z}_{\lambda-1}) = x_{n+1}y_{n+1}\left(\prod_{\ell=1}^{k_{n+1}-1} z_{\ell,n+1}\right)\partial E_n(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots, \mathbf{z}_{\kappa-2}),$$

where $\lambda = \max(\kappa - 1, k_{n+1})$, and ∂, as before, denotes the sum of all first-order partials (with respect to all variables in E_n). □

Theorem 3.4 is a special case of Theorem 3.5 with $k_i = r$, for all $1 \leq i \leq n$. Note that the diagonalized version of the polynomial defined in (20) need not be a symmetric function in all variables $x, y, z_1, z_2, \dots, z_{\kappa-2}$. Nevertheless, if we specialize all variables except x, i.e., by letting $y = z_1 = \cdots = z_{\kappa-2} = 1$ we get the following result of Brenti.

Theorem 3.6 (*Theorem 6.6.3 in [9]*)**.** *The descent generating polynomial over generalized Stirling permutations*

$$E_n(x) = \sum_{\sigma\in\mathcal{Q}_n^*} x^{\mathrm{des}(\sigma)}$$

has only real roots.

Another interesting generalization could be obtained using the urn model. Consider a scenario when instead of removing one ball, $s \geq 2$ balls are removed in each round. This way, we could define (r, s)-Eulerian numbers, polynomials, and investigate whether they are stable or not.

4. Further connections and open problems

4.1. Schröder and Legendre–Stirling numbers

Riordan in [32], along with devising the recurrence shown in (7) for the second-order Eulerian polynomials $C_n(x)$, mentions another related polynomial $T_n(x) = 2^n C_n(x/2)$, whose coefficients are related to the Schröder numbers. The recurrence

$$T_{n+1}(x) = (2n + 1)xT_n(x) + (2x - x^2)T_n'(x)$$

satisfied by these polynomials is almost identical to the one for $C_n(x)$, so $T_n(x)$ clearly seems susceptible to similar multivariate generalization. It would be interesting to study the combinatorial interpretation arising from such a multivariate refinement.

Egge defined the following family of permutations that arose while studying the Legendre–Stirling numbers (see Definition 4.5 in [15]). For each $n \geq 1$, let M_n denote the multiset

$$M_n = \{1, 1, \bar{1}, 2, 2, \bar{2}, \dots, n, n, \bar{n}\}$$

in which we have two unbarred copies of each integer j with $1 \leq j \leq n$ and one barred copy of each such integer. A *Legendre–Stirling permutation* π is a permutation of M_n such that if $i < j < k$ and $\pi_i = \pi_k$ are both unbarred, then $\pi_j > \pi_i$. A descent in a Legendre–Stirling permutation (which fits nicely with our definition of a descent) is a number i, $1 \leq i \leq 3n$, such that $i = 3n$ or $\pi_i > \pi_{i+1}$.

Now, let $B_{n,k}$ denote the number of Legendre–Stirling permutations of M_n with exactly k descents. Egge showed the following theorem.

Theorem 4.1 (*Theorem 5.1 of [15]*)**.** *For $n \geq 1$, the generating polynomial $\sum_{k=1}^{2n-1} B_{n,k}x^k$ has distinct, real, nonpositive roots.*

Finding a similar refinement (perhaps by defining auxiliary statistics, if needed) might lead to a better understanding of these permutations.

4.2. Stable q-analogues

Foata and Schützenberger introduced the following q-analog for Eulerian polynomials in [18]

$$A_n(x; q) = \sum_{\pi \in S_n} q^{\mathrm{cyc}(\pi)} x^{\mathrm{wex}(\pi)}, \tag{21}$$

where $\mathrm{wex}(\pi)$ counts the number of weak exceedances $\{i \mid \pi(i) \geq i\}$ and $\mathrm{cyc}(\pi)$ counts the number of cycles of the permutation π. Brenti [11] proved that this polynomial has only real roots when $q > 0$, and subsequently, Brändén [7] extended this result for the case when q was a negative integer. A special case of Proposition 4.4 in [8] for Ferrers boards of staircase shape, which gives a stability result for the α-permanent can be interpreted as the following q-analog of multivariate Eulerian polynomials (with $q = \alpha$). Thus, generalizing the above results of Brenti, in [8] essentially it was shown that

$$A_n(\mathbf{x}; q) = \sum_{\pi \in S_n} q^{\mathrm{cyc}(\pi)} \prod_{\pi_i \geq i} x_{\pi_i} \tag{22}$$

is stable when $q > 0$.

Another general form of q-analogues is given by the bivariate generating polynomials:

$$A_n^{\mathrm{M,E}}(\mathbf{x}; q) = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{stat_M}}(\pi) x^{\mathrm{stat_E}}(\pi), \tag{23}$$

where $\mathrm{stat_M}(\pi)$ refers to a Mahonian statistic, a statistic that is equidistributed with MacMahon's major index $\mathrm{maj}(\pi) = \sum_{\pi_i > \pi_{i+1}} i$ over permutations, and $\mathrm{stat_E}$ stands for an Eulerian statistic, a statistic that is equidistributed with the descents or weak exceedances. It would be interesting to see if a certain multivariate generating polynomial of Mahonian–Eulerian statistics, such as $\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{maj}(\pi)} \prod_{i \in \mathcal{D}(\pi)} x_{\pi_i}$, is stable. See [34] for various q- and (q, p)-analogs and recent unimodality results on their coefficients.

4.3. Durfee polynomials

Sagan and Savage showed that the Foata fundamental map ϕ over multiset permutations of $\{1^m, 2^n\}$ that can be interpreted as lattice paths has the following properties (see Corollary 2.4 in [33]). Let $\sigma \in \{1^m, 2^n\}$, then:

1. $\mathrm{maj}(\sigma) = \mathrm{inv}(\phi(\sigma)) = |\lambda|$,
2. $\mathrm{des}(\sigma) = \mathrm{durfee}(\lambda)$,

where $\mathrm{inv}(\sigma)$ denotes the number of inversions in the multiset permutation σ, λ is the partition cut out by $\phi(\sigma)$ – when viewed as a lattice path – from the $m \times n$ rectangle, $|\lambda|$ denotes the size of the partition, and $\mathrm{durfee}(\lambda)$ the size of the Durfee square of λ.

Simion proved that the Eulerian multiset polynomial has only real zeros [35], which in the light of the above corollary of Sagan and Savage immediately implies that the polynomial

$$\sum_{\lambda \in \mathcal{P}_{m,n}} x^{\mathrm{durfee}(\lambda)}$$

has real roots only, where $\mathcal{P}_{m,n}$ denotes the set of all partitions that fit in an $m \times n$ rectangle. Furthermore, from this corollary one can easily obtain a reformulation of the conjectures of Canfield et al. in [12] on the roots of Durfee polynomials. For example, they conjectured that the Durfee-generating polynomial over partitions of a fixed size, i.e., $\sum_{|\lambda|=n} x^{\mathrm{durfee}(\lambda)}$ has only real roots, which using the Foata bijection ϕ is equivalent to saying that the following restricted multiset Eulerian polynomial

$$\sum_{\substack{\sigma \in \{1^m, 2^m\} \\ \mathrm{maj}(\sigma) = m}} x^{\mathrm{des}(\sigma)} \tag{24}$$

has only real roots.

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