Math 581: Foundations of Combinatorics
Lecturer: Prof. Sara Billey

Problem Set #8

due Friday, December 8, 2017

Reading: Chapter 3, Section 3.1-3.8, plus take a look at "q-Rook placements and Jordan forms of upper-triangular nilpotent matrices" by Martha Yip.

Recommended Problems: Play with these problems before reading the solutions: Chapter 3, Problems 4, 5, 12, 15, 22.

Homework Problems: For each of the problems below, explain your answer fully. No credit will be given for a simple statement of the answer. Each problem is worth 10 points unless otherwise specified.

1. Exercise 45 (part (a)) from Chapter 3 of EC1.

2. Exercise 108 (parts (a) and (b)) from Chapter 3 of EC1.

3. Using a sign-reversing involution, prove that for all fixed $n > k$, the sum
$$\sum_{k \le m \le n} s(n,m)S(m,k)$$
equals zero, where $s(n,m)$ is the Stirling number of the first kind and $S(m,k)$ is the Stirling number of the second kind.

4. The $n \times n$ Vandermonde matrix has entries of the form x_i^{j-1} for $1 \le i,j \le n$. The Vandermonde determinant can be written two ways
$$\prod_{1 \le i < j \le n} (x_j - x_i) = \sum_{w \in S_n} (-1)^{inv(w)} x_1^{(w_1-1)} x_2^{(w_2-1)} \cdots x_n^{(w_n-1)}.$$
Give a direct proof of this polynomial identity by an explicit involution.

5. Let $(X, \le)$ be a finite poset. If r is the size of the largest chain in X, then show that X can be partitioned into r antichains.

6. Describe the covering relation for Bruhat order on $\mathfrak{S}_n$.

7. How many maximal chains does the partition lattice Π_n have?

8. Show that an interval in the Boolean lattice B_n is again a Boolean lattice.

9. Consider the partial order on partitions ordered by containment, i.e. $\lambda \le \mu$ if $\lambda = (\lambda_1, \lambda_2, \dots)$, $\mu = (\mu_1, \mu_2, \dots)$, and each $\lambda_i \le \mu_i$. So if $\lambda \le \mu$ then the Ferrers diagram for μ contains the Ferrers diagram for λ. Which of the 10 nice properties does this poset have?

10. Restrict the partial order on partitions above to just the partitions which sit inside an $n \times k$ box. What nice properties does this poset have?

11. (Bonus) Prove that the partition lattice Π_n is unimodal.