Math 561: Foundations of Combinatorics
Lecturer: Prof. Sara Billey

Problem Set #7

due Wednesday, November 22, 2017

Reading: In Chapter 2, read Section 2.1-2.4.

Recommended Problems: Play with these problems before reading the solutions: EC1 Chapter 2. Chapter 2, Chapter 2, Problems 3,4,6,8, 11, 22.

Homework Problems: For each of the problems below, explain your answer fully. No credit will be given for a simple statement of the answer. Each problem is worth 10 points unless otherwise specified.

1. Exercise 2 from Chapter 2 of EC1.

2. Exercise 10 from Chapter 2 of EC1.

3. Exercise 19 from Chapter 2 of EC1.

4. How many ways can one rearrange $2n$ people seated around a circular dinner table so that no one faces the same person they were previously facing? Compute the answer explicitly for $n = 4$.

5. Prove that for any $T \subset [n-1]$, the number of permutations in S_n with descent set exactly T is given by a polynomial function in n. What is the degree of the polynomial in terms of T?

6. Show that the number of permutations with exactly 1 small descent equals the number of derangments in S_n. A small descent is an index i such that $w_i = w_{i+1} + 1$.

7. How many permutations in $\mathfrak{S}_n$ have no cycles of length k? What fraction of all permutations in $\mathfrak{S}_n$ does this represent as n goes to infinity?

8. Let $f(m, n)$ be the number of $m \times n$ matrices of 0's and 1's with at least one 1 in every row and column. Show that

$$f(m,n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2^{n-k} - 1)^m.$$

9. (Bonus) Use Burnside's Lemma to derive Jordan's theorem that if a group acts transitively on a set then it has a derangement (a fixed point free element).