Math 561: Foundations of Combinatorics
Lecturer: Prof. Sara Billey

Problem Set #3

due Wednesday, October 18, 2017

Reading: In Chapter 1, read Sections 1.4 - 1.7. Plus, read "On the enumeration of tanglegrams and tangled chains" by Billey, Konvalinka and Matsen, published in Journal of Combinatorial Theory Series A, 2017. Available on class website under "Other Handouts".

Recommended Problems: Give each of these problems careful consideration before reading the solutions: Chapter 1: 58, 61, 62, 116, 124.

Homework Problems: For each of the problems below, explain your answer fully. No credit will be given for a simple statement of the answer. Each problem is worth 10 points unless otherwise specified.

1. Exercise 118(a) from Chapter 1 of EC1. Check out parts (b) and (c) too, but no need to write them up.

2. Use string diagrams of permutations to find a q-analog of the Fibonacci numbers and a combinatorial interpretation of the coefficients.

3. How many nested sequences of subsets are there of the form

$$(T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k)$$

with $T_i \subseteq [n]$ and $k \in \mathbb{P}$?

4. Prove the following formula for the (signless) Stirling number of the first kind $c(n, 2)$:

$$c(n,2) = (n-1)!\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}\right) .$$

5. A perfect shuffle is done by taking a deck of cards, splitting it exactly in half, putting the first half in the right hand, the second half in the left hand and then dropping one card from left hand, then one from the right, then one from the left etc.

   (a) What is the cycle notation for this permutation of the deck of 52 cards?

   (b) How many times must one do a perfect shuffle before the deck returns to its original order?

6. The names of 100 contestants are placed in 100 wooden boxes, one name to a box, and the boxes are lined up on a table in a room. One by one, the contestants are led into the room; each may look in at most 50 boxes, but must leave the room exactly as she/he found it and is permitted no further communication with the others.

   The contestants have a chance to plot their strategy in advance, and they are going to need it, because the team of contestants win the prize only if every single contestant finds his/her own name. Find a strategy for them which which has probability of success exceeding 30%.

7. Prove that 231-avoiding permutations are Catalan objects.

8. For a fixed $w \in S_n$, find formula for the number of permutations $v \in S_n$ such that $vwv^{-1} = w$.

9. Bonus: What is the expected number of braids in a reduced expression for $w_0 \in S_n$. A braid is three consecutive generators of the form $s_i s_{i+1} s_i$ or $s_{i+1} s_i s_{i+1}$.