Math 561: Foundations of Combinatorics
Lecturer: Prof. Sara Billey

Problem Set #1

due Wednesday, October 4, 2017

Reading: In Chapter 1, read Sections 1.1 and 1.2 specifically omitting pages 10-14 for now. Also read the Appendix on Graph Theory Terminology pages 571–573.

Recommended Problems: Give each of these problems careful consideration before reading the solutions: Chapter 1: 2, 3ab, 8b, 9a.

Homework Problems: For each of the problems below, explain your answer fully. No credit will be given for a simple statement of the answer. Each problem is worth 10 points unless otherwise specified.

1. Use the Pigeonhole Principle to prove that the decimal expansion of any rational number m/n eventually is repeating.

2. (a) Prove combinatorially $\binom{n}{1} + 6\binom{n}{2} + 6\binom{n}{3} = n^3$.

   (b) Prove combinatorially $\binom{0}{k} + \binom{1}{k} + ... + \binom{n}{k} = \sum_{j=0}^{n} \binom{j}{k} = \binom{n+1}{k+1}$ where n, k are any fixed non-negative integers.

   (c) Find a simple form for $1^3 + 2^3 + \cdots + n^3$.

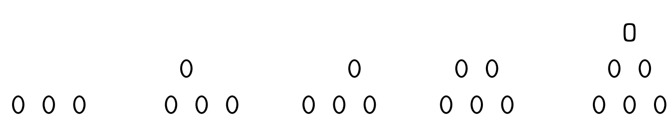
3. Find the generating function for each of the following sequences and restate the problems in combinatorial language.

   (a) The number of collections of n balloons with a multiple of three red ones, at least 2 blue ones and no more than 5 green ones. Assume balloons of the same color are indistinguishable.

   (b) The number of ways one can give n children each one balloon such that the entire set of balloons has a multiple of three red ones, at least 2 blue ones and no more than 5 green ones.

4. Prove that for any $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$ there exist two of the integers a_i and a_j with $i \neq j$ such that $a_i - a_j$ is divisible by n.

5. How many ways can coins be stacked in the plane such that the bottom row consists of n consecutive coins? For example, there are five ways to do this if $n = 3$:



6. Find an explicit formula for the number of cards in a card house of depth 1 and with n levels. For example, it takes 2 cards to make a small house of one level built by leaning them against each other. To build a second level, build two small houses together and put a top on it across both peaks. Then build a second small house on top.

7. (30 pts) Prove any 3 parts to Problem 35 in Chapter 1 of EC1 rated 2 or 2+. Try each part before consulting the solutions. Additional parts count as bonus problems for additional credit.

8. Let a_n be the number of ways to write the positive integer n as the sum of powers of 2, disregarding order.

   (a) Find a_n for $n = 1, 2, 3, \ldots, 10$

   (b) Give a generating function for a_n.

   (c) Use your answers in parts a) and b) to conjecture and prove a result comparing a_{2n} and a_{2n+1}.

9. (Bonus (20 pts)): Let b_n be the number of different rooted binary trees with n unlabeled leaves. Show

$$b_n = \sum_{\lambda} \frac{\prod_{i=2}^{\ell(\lambda)} (2(\lambda_i + \cdots + \lambda_{\ell(\lambda)}) - 1)}{z_\lambda},$$

where the sum is over all decreasing sequences $\lambda = (\lambda_1, \ldots, \lambda_k)$ which sum to n and each part λ_i is a power of 2. The notation $\ell(\lambda) = k$ is the number of nonzero parts of λ, and

$$z_\lambda = (1^{m_0} 2^{m_1} 4^{m_2} 8^{m_3} \cdots)\,(m_0! m_1! m_2! \cdots)$$

if λ has m_i parts equal to 2^i for all integers $i \leq 0$.