MATH 340A: Homework 6

Due on Gradescope by **August 11th** at 11:59pm.

Problem 1. Let $V = \mathbb{R}[x]_{\leq 2}$ be the $\mathbb{R}$-vector space of polynomials with degree at most 2, and fix a basis $\{1, x, x^2\}$ for V. Define a map $B : V \times V \to \mathbb{R}$ by

$$B(f, g) = \int_0^1 f(t)g(t)\,dt$$

(a) Show that B is a bilinear form.

(b) Find the matrix associated to B.

(c) Is B nondegenerate?

Problem 2. Let M be an $n \times n$ symmetric matrix. Show that M is positive definite (i.e. $v^T M v > 0$ for all $v \neq 0$) if and only if all eigenvalues of M are positive. You may (and should!) use the result of the Spectral Theorem.

Problem 3. Recall from class that for an inner product space V and subspace $W \subset V$, we can define the orthogonal complement of W to be

$$W^\perp = \{v \in V \mid \langle v, w\rangle = 0 \ \ \forall\ w \in W\}$$

Prove the following statements about orthogonal complements.

(a) Show that $W^\perp$ is a subspace of V.

(b) Show that $V = W \oplus W^\perp$.

(c) Show that $W^\perp \cong V/W$.

(d) Show that $\dim(W^\perp) = \dim(V) - \dim(W)$.