MATH 340A: Homework 5

Due on Gradescope by **July 28th** at 11:59pm.

Problem 1. (a) Suppose F is algebraically closed, and V is a non-zero finite-dimensional F-vector space. Let $T : V \to V$ be a linear map. Show that T has an eigenvector.

(b) Give an example of an infinite-dimensional V and linear map $T : V \to V$ which does not have an eigenvector.

(c) Suppose V is a finite-dimensional F vector space, and $T : V \to V$ is an invertible linear map. Describe the eigenvectors and eigenvalues of T^{-1} in relation to those of T.

Problem 2. Let V be an F-vector space of dimension n (F is not necessarily algebraically closed), and let $T : V \to V$ be a linear map. Show that if the characteristic polynomial of T is given by

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

then $\det(T) = \pm a_0$ and $\mathrm{tr}(T) = \pm a_{n-1}$. Additionally, for each case, describe when the sign is $+$ and when the sign is $-$.

Problem 3. Finish the proof of the Cayley-Hamilton Theorem from class, using the following argument.

(a) Show that if $p(t) \in F[t]$ is a polynomial, and M is a matrix with entries in F, then determining if $p(M) = 0$ is independent of basis (i.e. if $M' = P^{-1}MP$, then $p(M') = 0$ if and only if $p(M) = 0$).

(b) For a given matrix M with entries in F, think of M as a matrix with entries in $\overline{F}$, and change M to Jordan Normal Form. Conclude that $p(M) = 0$, proving the theorem.

Problem 4. Let S denote the set of 4×4 matrices over $\mathbb{C}$ whose eigenvalues are a subset of the set $\{a, b\}$ with $a \neq b$. Determine a full set of similarity representatives for S. In other words, provide a set of matrices S' such that any matrix in S is similar to exactly one matrix in S'.

Problem 5. Let V be a finite-dimensional F-vector space, and let F be algebraically closed. Let $T : V \to V$ be a linear map. Then, the **minimal polynomial** of T is the monic (i.e. leading coefficient is 1) polynomial $m(t) \in F[t]$ of lowest degree such that $m(T) = 0$.

(a) Show that $m(t)$ is unique. In other words, if $m_1(t)$ and $m_2(t)$ are both minimal polynomials for T, then $m_1(t) = m_2(t)$.

(b) Show that if $q(t)$ is a polynomial such that $q(T) = 0$, then $m(t)$ divides $q(t)$ (You may need to assume polynomial division/remainder theorem). Conclude that since the characteristic polynomial $p(t)$ satisfies $p(T) = 0$ by Cayley-Hamilton, we must have that $m(t)$ divides $p(t)$, so the factors of $m(t)$ are precisely $(t - \lambda_i)$ raised to some power.

(c) Using Jordan Normal Form, show that if $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of T, then the exponent of $(t - \lambda_i)$ in $m(t)$ is the size of the largest Jordan block associated to λ_i. Conclude that if the minimal polynomial has no repeated roots, then T is diagonalizable.

(d) Using the argument from problem 3, show that if F is not algebraically closed, the above results still hold.