MATH 340A: Homework 4

Due on Gradescope by **July 21st** at 11:59pm.

Problem 1. Let V, W be finite-dimensional F-vector spaces, let $T : V \to W$ be a linear map, and fix bases for V and W. Prove that the matrix representation of the dual map $T^* : W^* \to V^*$ (in the dual bases for V and W) is the transpose of the matrix representation for T. (If this problem is too tedious, feel free to just prove the case where $W = V$).

Problem 2. Recall from class the definitions of $\mathrm{Sym}^k(V)$ and $\mathrm{Alt}^k(V)$ for a finite dimensional F-vector space V.

(a) Prove that $\mathrm{Sym}^k(V)$ is a subspace of $\mathrm{Hom}_F(V^k; F)$ of dimension $\binom{n+k-1}{k}$.

(b) Prove that $\mathrm{Alt}^k(V)$ is a subspace of $\mathrm{Hom}_F(V^k; F)$ of dimension $\binom{n}{k}$.

Problem 3. Let $\mathrm{GL}_n(F)$ be the set of invertible $n \times n$ matrices with entries in F. Determine the number of elements in $\mathrm{GL}_n(\mathbb{F}_p)$ (as a function of p and n).

Problem 4. Let $\alpha_1, \ldots, \alpha_n \in F$. Then, the Vandermonde matrix is given by

$$V(\alpha_1, \ldots, \alpha_n) = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \cdots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix}$$

Prove that

$$\det(V(\alpha_1, \ldots, \alpha_n)) = \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)$$

using the following steps.

(a) Let $V = F[x]_{\le(n-1)}$ and let $W = F^n$. Choose bases $B_V = \{1, x, \ldots, x^{n-1}\}$ and $B_W = \{e_1, \ldots, e_n\}$. Find a linear map $T : V \to W$ whose matrix representation is $V(\alpha_1, \ldots, \alpha_n)$. Describe the map without using bases, and prove that the map is indeed linear.

(b) Show that the basis

$$\begin{aligned} B'_V &= \left\{ \prod_{i=1}^{j} (x - \alpha_i) \;\middle|\; 0 \le j \le n-1 \right\} \\ &= \{1, (x - \alpha_1), (x - \alpha_1)(x - \alpha_2), \ldots, (x - \alpha_1) \cdots (x - \alpha_{n-1})\} \end{aligned}$$

is a basis for V.

(c) Find the matrix representation for T in the bases B'_V and B_W, and compute $\det(T)$ in this basis instead (Hint: it should be easy to compute $\det(T)$ in this basis, because the matrix representation has a special property!).

Problem 5. Recall that $\mathbb{C}$ is an $\mathbb{R}$-vector space of dimension 2.

(a) Fix $\lambda \in \mathbb{C}$. Show that multiplication by λ, defined by the function

$$\begin{aligned} m_\lambda : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto \lambda z \end{aligned}$$

is an $\mathbb{R}$-linear transformation from $\mathbb{C}$ to $\mathbb{C}$.

(b) Suppose $\lambda = 1 + i$. Using the basis $\{1, i\}$ for $\mathbb{C}$, find the matrix representation M_λ of λ.

(c) Suppose $\lambda = 1 + i$. Using the basis $\{1, 2 + i\}$ for $\mathbb{C}$, find the matrix representation for M_λ of λ.

(d) Suppose $\lambda = 1 + i$. Determine $\det(m_\lambda)$.

(e) Suppose $\lambda = 1 + i$. Find a polynomial $p(z)$ such that $p(\lambda) = 0$. Compute $p(M_\lambda)$ (that is, evaluate the polynomial $p(z)$ where z is replaced by the matrix M_λ). Explain why this result should be expected (does not have to be rigorous).