MATH 340A: Homework 3

Due on Gradescope by **July 14th** at 11:59pm.

Problem 1. Let V, W be finite-dimensional F-vector spaces, and let $T : V \to W$ be an invertible linear map. Show that $T^{-1} : W \to V$ is also a linear map.

Problem 2. Let V, W be finite-dimensional F-vector spaces, and let $T : V \to W$ be a linear map.

(a) Show that T is injective if and only if it takes linearly independent sets of V to linearly independent sets of W.

(b) Show that T is surjective if and only if it takes spanning sets of V to spanning sets of W.

(c) Show that the composition of two injective, surjective, or bijective maps is injective, surjective, or bijective, respectively.

(d) Let $T = T_1 \circ T_2$ be the composition of linear maps T_1 and T_2. If T is bijective, can you say anything about T_1 or T_2? Specifically, does T_1 or T_2 have to be injective, surjective, or bijective?

Problem 3. Let V, W be finite-dimensional F-vector spaces, and let $T : V \to W$ be a linear map.

(a) Prove that if $T : V \to W$ is injective, then $\dim(V) \leq \dim(W)$. Similarly, prove that if $T : V \to W$ is surjective, then $\dim(V) \geq \dim(W)$.

(b) Prove that if $\dim(V) = \dim(W)$, then the following are equivalent:

- T is injective.
- T is surjective.
- T is invertible.

(c) Suppose $V' \subset V$ is a subspace, and $\dim(V') = \dim(V)$. Show that $V' = V$.

Problem 4. Let V be a finite-dimensional F-vector space, and let $T : V \to V$ be a linear map. Suppose $T^2 = T$.

(a) Show that $V = \ker(T) \oplus \operatorname{im}(T)$.

(b) Show that the reverse implication is not true. In other words, give an example of a linear map $T : V \to V$ with $V = \ker(T) \oplus \operatorname{im}(T)$ but $T^2 \neq T$.

Problem 5. Let U, V, and W be finite-dimensional F-vector spaces. We say that a **short exact sequence** of vector spaces is given by

$$\{0\} \longrightarrow U \xrightarrow{S} V \xrightarrow{T} W \longrightarrow \{0\}$$

where $S : U \to V$ and $T : V \to W$ are linear maps, S is injective, T is surjective, and furthermore $\mathrm{im}(S) = \ker(T)$ as subspaces of V. Prove the following statements about short exact sequences.

(a) If $U = \{0\}$ or $W = \{0\}$, then $U \cong V$ or $V \cong W$, respectively.

(b) Prove that $\dim(V) = \dim(U) + \dim(W)$.

(c) Let V be a finite-dimensional F-vector space, and let $W \subset V$ be a subspace. Show that

$$\{0\} \longrightarrow W \xrightarrow{i} V \xrightarrow{\pi} V/W \longrightarrow \{0\}$$

is a short exact sequence, where i represents the inclusion of W into V, and $\pi : V \to V/W$ is the linear map which sends $v \in V$ to $[v] \in V/W$.

(d) Show that in a short exact sequence of finite-dimensional vector spaces, we have that $V \cong U \oplus W$. Do not use a dimension-counting argument!