MATH 340A: Homework 2

Due on Gradescope by **July 7th** at 11:59pm.

Problem 1. Let $V = F[x]$, and for $n \geq 0$, let W_n be the set of polynomials divisible by x^n, or mathematically

$$W_n = \{f(x) \in F[x] \, : \, f(x) = x^n g(x), g(x) \in F[x]\}$$

(a) Show that W_n is a subspace of V for all $n \geq 0$.

(b) Show that W_n is a subspace of W_m whenever $n \geq m$ ($n, m \geq 0$).

(c) Let $W_i' = W_i \cap F[x]_{\leq n}$. Show that $F[x]_{\leq n}$, the set of polynomials with coefficients in F of degree at most n, is given by

$$F[x]_{\leq n} = W_0' + W_1' + \cdots + W_n'$$

Is this sum direct?

Problem 2. Let V be a vector space over F, and let $W \subset V$ be a subspace.

(a) Show that for $v, w \in V$, the relation $v \sim w$ if $v - w \in W$ is indeed an equivalence relation (i.e., satisfies reflexivity, symmetry, and transitivity).

(b) Let $[x] = \{y \in V \, : \, x \sim y\}$ denote the equivalence class of x. Show that for $v, w \in V$, either $[v] = [w]$ or $[v] \cap [w] = \emptyset$. Conclude that the equivalence classes under $\sim$ partition V (i.e. every element of V is in exactly one equivalence class).

(c) Show that the operations defined by

$$[v] + [w] = [v + w] \qquad c[v] = [cv]$$

are well-defined. In other words, show that if v is replaced by v' such that $v \sim v'$ (and similarly for w), the operations output the same equivalence class.

(d) Show that the additive identity of V/W, denoted $\vec{0}_{V/W}$, is given by $[\vec{0}_V]$. Furthermore, show that if $w \in W$, then $[w] = [\vec{0}_V]$.

(e) Assume that V is finite dimensional. Show that $\dim(V/W) = \dim(V) - \dim(W)$. (Hint: pick a basis for W, and extend to a basis for V. Then, take the equivalence class of each of these basis vectors in V/W, and prove that the non-zero classes form a basis for V/W.

Problem 3. Let $V = C(\mathbb{R})$, the vector space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Note that the additive identity in V is given by the function $f(x) = 0$.

(a) Show that e^x and e^{-x} are linearly independent in V.

(b) Give an example of three vectors $f, g, h \in V$ which are linearly dependent, but any two of them are linearly independent. Justify your answer!

Problem 4. Prove the following statement, or give a counterexample: if $\{v_1, \ldots, v_n\}$ are linearly independent, and $\{w_1, \ldots, w_n\}$ are linearly independent, then $\{v_1 + w_1, \ldots, v_n + w_n\}$ is linearly independent.

Problem 5. Consider $F[x]$, the vector space (over F) of polynomials with coefficients in F. Show that $F[x]$ does not have a finite basis, and thus is infinite dimensional. Note that it is not enough to show that any one basis is infinite, since the theorem proven in class which states that every basis has the same size only applies to finite bases.