MATH 340A: Homework 1

Due on Gradescope by **June 30** at 11:59pm.

Problem 1. Prove the following facts about fields. Let F be a field.

(a) Show that the additive and multiplicative identity of F are both unique.

(b) Show that the additive and multiplicative inverses in F are both unique.

(c) Show that for $a, b \in F$, if $ab = 0$, then either $a = 0$ or $b = 0$.

(d) Define the set
$$\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}.$$
Show that $\mathbb{Q}(\sqrt{2})$ is a field.

Problem 2. Prove the following facts about finite fields. Let F be a finite field.

(a) Let $\mathbb{F}_n$ be the integers modulo n, as constructed in class. Show that if $\mathbb{F}_n$ is a field, then n must be prime.

(b) Show that $\text{char}(F) > 0$.

(c) Show that F cannot be algebraically closed. Conclude that if K is an algebraically closed field, K must have infinitely many elements.

(d) Construct an addition and multiplication table for a field with 4 elements, or prove that one cannot exist. If one exists, what is its characteristic?

Problem 3. For each of the following, either verify that it is a vector space (over the given field), or prove that it is not.

(a) $\mathbb{R}^2$, with the usual addition operation, but $c(x, y) = (cx, 0)$ for $c \in \mathbb{R}$ (over the field $\mathbb{R}$).

(b) The set of differentiable functions $f : \mathbb{R} \to \mathbb{R}$, with $f'(0) = f'(1) = 1$, under usual addition and multiplication (over $\mathbb{R}$).

(c) $\mathbb{F}_2^2$ under the usual multplication, but addition is replaced by
$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a - b \\ c - d \end{pmatrix}$$
(over $\mathbb{F}_2$).

Problem 4. Let V be a vector space over the field F.

(a) Prove that for $c \in F$ with $c \neq 0$, and $v, w \in V$, if $cv = cw$, then $v = w$.

(b) Prove that for $a, b \in F$ and $v \in V$ $(v \neq 0)$, if $av = bv$, then $a = b$.

(c) Prove that for $a \in F$, $a0 = 0$ (where 0 denotes $0 \in V$).

(d) Prove that for $c \in F$ and $v \in V$, if $cv = 0$, then either $c = 0 \in F$ or $v = 0 \in V$.

Problem 5. Give an example of a nonempty subset $S \subset \mathbb{R}^n$ which is **not** a vector space, but is...

(a) ...closed under vector addition.

(b) ...closed under scalar multiplication (by $\mathbb{R}$).

(c) ...closed under taking additive inverses.