This will be a journey into algebraic topology, using linear algebra as a vital tool in making calculations.

Algebraic Topology tries to answer questions in topology (geometric questions) by making the questions more algebraic.

One of the main questions we ask:

<u>Q:</u> How many holes does my space have?

We are going to give a way to answer this question and investigate some machinery of algebraic topology using vector spaces and linear algebra.

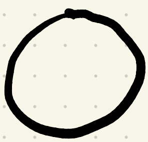
Topology

This is the study of abstract spaces up to continuous deformation

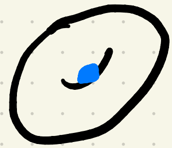
(No algebraic structure + we can wiggle and smush them around)

Spaces of interest

- The circle

$S^1 :=$ 

- The torus

$T^2 :=$ 

to a topologist, this is the same as a coffee mug!

- The Klein bottle

- The sphere $S^2 :=$ 

- The disk

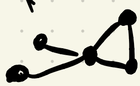
D' 

- The Ball

- The mobius band

- The cylinder

- graphs



The point here is that I could go on and on; unlike a vector space, which has a very strict definition, topological spaces can look very crazy!

Q: How can we classify topological spaces?

One answer: Introduce some linear algebra! (to capture the topology)

If we fix a field F, then one way to distinguish between vector spaces is by their dimension. If $\dim V \neq \dim W$, then $V \ncong W$.

Let's try to systematically associate some vector spaces to a space X so that we can use them as invariants! This will give us a way to distinguish spaces.

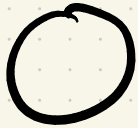
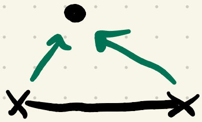
But first...

Combinatorics?

Let's try to build our spaces in a "blue print" way, out of vertices, line segments, triangles and their higher generalizations

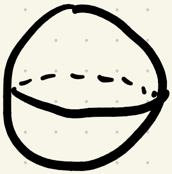
(turns out, we can do this for most nice spaces!)

Ex:

$S^1 =$  $=$ 

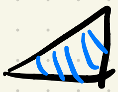
1 vertex

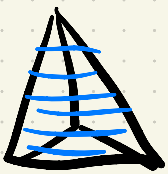
1 edge

$S^2 =$  $=$ 

1 vertex

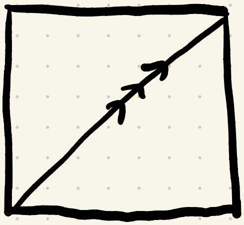
1 face

$D^1 \simeq$  1 face

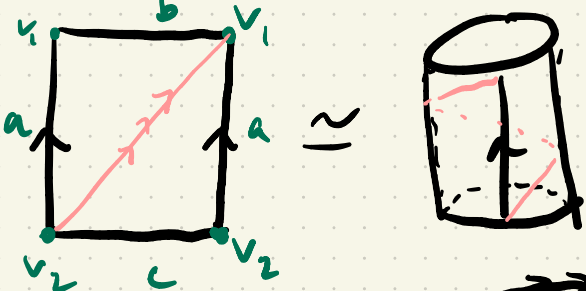
$D^2 \cong$  1 tetrahedron (face + another dimension)

We see that we can "glue" these things together. What if we take this further?

Let's start with "a square". What happens if we glue some of the sides together?



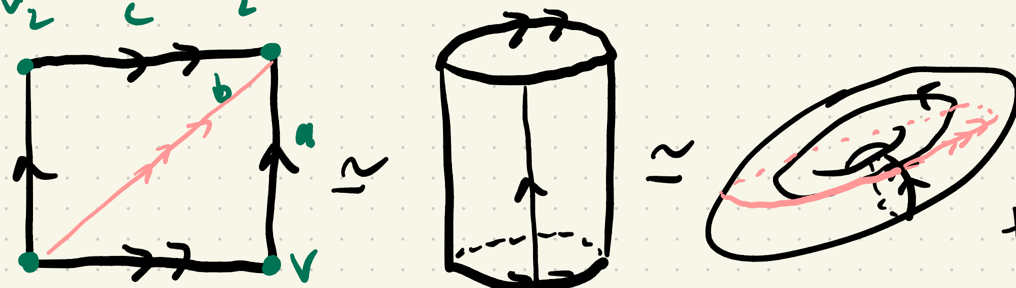
Ex!



cylinder!

2 vertices

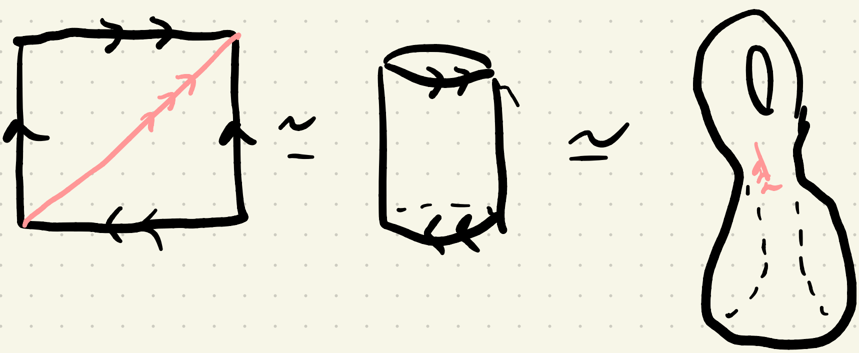
4 edges



1 vertex

3 edges

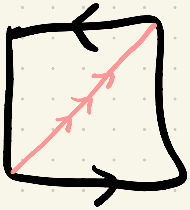
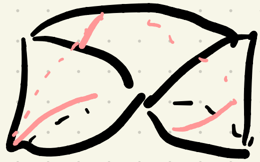
torus!



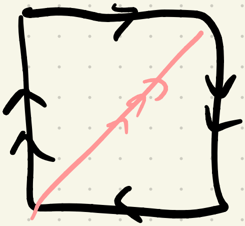
Klein Bottle!

1 vertex

3 edges

 $\simeq$  Mobius!

2 vertices
4 edges

 $\simeq$??

We call this $\mathbb{RP}^2$

1 vertex
3 edges

Notice that when we glue, we identify edges & vertices.

In fact, this is what we need to create our invariant! These are called cell complexes; the data of their cells (vertices, edges, faces) and how they're attached is all we need:

Cellular Homology

The idea: quantify "holes" in my space.

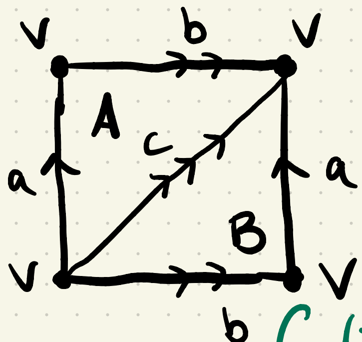
Let X be a cell complex. Suppose:

- X has 0-cells $\{v_1, \dots, v_i\}$ (vertices)
- X has 1-cells $\{e_1, \dots, e_j\}$ (edges)
- X has 2-cells $\{f_1, \dots, f_k\}$ (faces)

$\vdots$

Define for each $n \geq 0$ the vector space $C_n(X)$, the n-chains of X, as the $\mathbb{R}$-vector space with one basis element for each n-cell.

Ex:

Let $X = T^2 =$ 

$C_1(T^2) = \mathbb{R}\{a, b, c\} \cong \mathbb{R}^3$

Then

$C_0(T^2) = \mathbb{R}\{v\} \cong \mathbb{R}$, $C_2(T^2) = \mathbb{R}\{A, B\} \cong \mathbb{R}^2$

For each $n \geq 0$ there is a linear transformation

$$C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$$

(specify on a basis!)

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{\hat{v}_i}$$

what this means is we take an n-cell σ and look at all the (n-1)-cells inside and take their alternating sum.

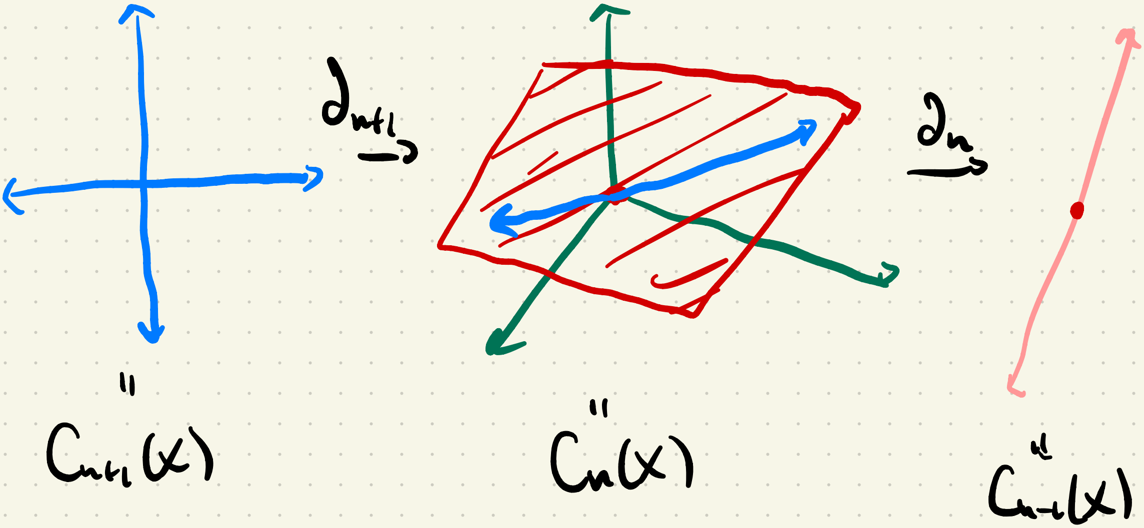
So, we have a sequence of linear maps:

$$\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

$$\cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0$$

Fact:

$\operatorname{im} \partial_{n+1}$ is a subspace of $\ker \partial_n$



So, we can always form the quotient vector space!!

$$\frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = H_n(X) \leftarrow \text{n-th homology of } X$$

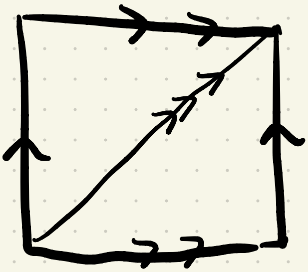
Fact

If $X \cong Y$ as topological spaces, then $H_n(X) \cong H_n(Y)$ $\forall n \geq 0$

☆ To tell spaces apart, compute homology! ☆

Ex:

Back to the torus:



$$\cdots \to 0 \to C_2(T^2) \xrightarrow{\partial_2} C_1(T^2) \xrightarrow{\partial_1} C_0(T^2) \longrightarrow 0$$

Immediately, we know that for $n \geq 3$, $H_n(T^2) = 0$.

$H_2(T^2)$

This is $\ker\partial_2 / \operatorname{im}\partial_3 = \ker\partial_2$

since $\operatorname{im}(\partial_3 : 0 \to C_2(T^2)) = 0$.

$C_2(T^2) = \mathbb{R}^2$, $\quad C_1(T^2) = \mathbb{R}^3$,

so we can realize this as a matrix $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$!

Just need to know the value on the basis.

$\partial_2(A) = a + b - c = \partial_2(B)$

$\rightsquigarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$ rank $= 1$, so nullity $= 1$

$\rightsquigarrow \ker\partial_2 \cong \mathbb{R} \cong H_2(T^2)$

$H_1(T_2)$

This is $\ker\partial_1 / \mathrm{im}\,\partial_2 = \ker\partial_1 / \overbrace{\langle a+b-c\rangle}^{\text{1-dim}}$.

What is $\ker\partial_1$?

Find the matrix.

$\partial_1(a) = \underset{\substack{\| \\ v-v}}{0}$, $\quad \partial_1(b) = \underset{\substack{\| \\ v-v}}{0}$ $\quad \partial_1(c) = \underset{\substack{\| \\ v-v}}{0}$

$\rightsquigarrow \ker\partial_1 = \mathbb{R}^3$ (everything!)

Thus, $H_1(T^2) \cong \mathbb{R}^2$

$H_0(T^2)$

$\ker\partial_0 / \mathrm{im}\,\partial_1 = \mathbb{R}/0 = \mathbb{R}$.

Cool!

Let's look at some other spaces. If they have __any__ different homology, then they are not the same space!

__Ex:__

 $= S^1$

$C_0(S^1) \cong \mathbb{R}\{v\}$ $C_1(S^1) \cong \mathbb{R}\{e\}$

$H_n(S^1) = 0 \quad \forall n \geq 2$

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{R}\{e\} \xrightarrow{\partial_1} \mathbb{R}\{v\} \longrightarrow 0$$

__$H_1(S^1)$__

$= \ker \partial_1 / \operatorname{im} \partial_2 = \ker \partial_1$

$\partial_1(e) = v - v = 0 \quad \leadsto \ker \partial_1 = \mathbb{R}$

so $H_1(S^1) = \mathbb{R}$

__$H_0(S^1)$__

$= \ker \partial_0 / \operatorname{im} \partial_1 = \mathbb{R}\{v\}/0 \cong \mathbb{R}$

So $S^1 \not\cong T^2$! These are "different" to the eyes of topology.

Ex:

$$C_0(S^2) = \mathbb{R}\{v\} \qquad C_1(S^2) = 0$$

$$C_2(S^2) = \mathbb{R}\{f\}$$

$$\cdots \longrightarrow \mathbb{R}\{f\} \xrightarrow{\partial_2} 0 \xrightarrow{\partial_1} \mathbb{R}\{v\} \xrightarrow{\partial_0} 0$$

$$H_n(S^2) = \begin{cases} \mathbb{R} & n = 0, 2 \\ 0 & \text{else.} \end{cases}$$

Different again!

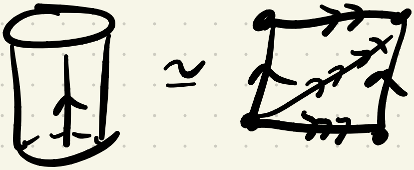
Ex:

$$C_2(D^2) = \mathbb{R}\{f\}$$

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{R}\{f\} \xrightarrow{\partial_2} 0 \longrightarrow \cdots$$

$$H_n(D^2) = \begin{cases} \mathbb{R} & n = 2 \\ 0 & \text{else} \end{cases}$$

Ex:



$X = $ cylinder

$C_0(X) = \mathbb{R}\{u, v\}$

$C_1(X) = \mathbb{R}\{a, b, c, d\}$

$C_2(X) = \mathbb{R}\{f_1, f_2\}$

$H_n(X) = 0$ for $n \geq 3$

$0 \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$

$H_2(X)$

$\ker \partial_2$

$\partial_2(f_1) = b - d + a$, $\partial_2(f_2) = a - d + c$

both of these are linearly ind. in $C_1(X)$

$\leadsto H_2(X) = 0$

$H_1(X)$

$\ker \partial_1 / \operatorname{im} \partial_2$

$\operatorname{im} \partial_2 = \mathbb{R}\{b - d + a, a - d + c\} \cong \mathbb{R}^2$

$\partial_1(a) = w - v$, $\partial_1(b) = 0$, $\partial_1(c) = 0$

$\partial_1(d) = w - v$

$\rightsquigarrow \operatorname{im}\partial_2 = \mathbb{R}\{w-v\} \cong \mathbb{R}$

$\rightsquigarrow \ker\partial_2 \cong \mathbb{R}^3$ (Rank-Nullity)

$\rightsquigarrow H_1(X) \cong \mathbb{R}^3/\mathbb{R}^2 = \mathbb{R}$

$\underline{H_0(X)}$

$\mathbb{R}^2/\mathbb{R} = \mathbb{R}$ Same as circle!

In fact, $S^1 \simeq X$ (homotopy eq.)