Math 224: Advanced Multivariable Calculus

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1 Double Integrals

Textbook Reference: Chapter 15.1.
Planned Lecture Date(s): March 27, 2023.

Recall from a first year calculus course that we can approximate the area under a curve $y = f(x)$ on an interval $[a, b]$ by subdividing $[a, b]$ into n equally sized subintervals and choosing a sample point x_i within each subinterval. We then approximate using the Riemann sum

$$\text{Area} \approx \sum_{i=1}^{n} f(x_i)(\Delta x)$$

where $\Delta x = \frac{b-a}{n}$. This can be represented geometrically as taking a rectangle of height $f(x_i)$ and width Δx on top of each subinterval, and summing the total areas of the rectangles.

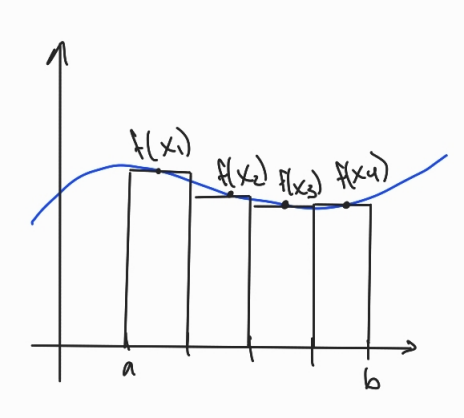


Figure 1: Approximating the area under the blue curve via a Riemann sum.

Taking a limit as $n \to \infty$, we can define the integral of f on the interval $[a, b]$ as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(\Delta x).$$

The integral then geometrically represents the area between the curve $y = f(x)$ and the x-axis. Note that this area is "signed", in that rectangles below the x-axis are taken to have negative area (product of positive width and negative height), and the integral keeps track of this on our behalf. For curves where $f(x) > 0$ for all $x \in [a, b]$, this agrees with our usual notion of area.

We could do a similar construction with a function $z = g(x, y)$ of two variables, defined on a rectangle $[a, b] \times [c, d]$. Graphing this as a surface, we could approximate the volume under the surface by first subdividing $[a, b] \times [c, d]$ into subrectangles. We then sum the volume of the rectangular prisms with base $(\Delta A) = (\Delta x)(\Delta y)$ and height $f(x_i, y_j)$, for some choice of sample point (x_i, y_j) in the (i, j)-th subrectangle.

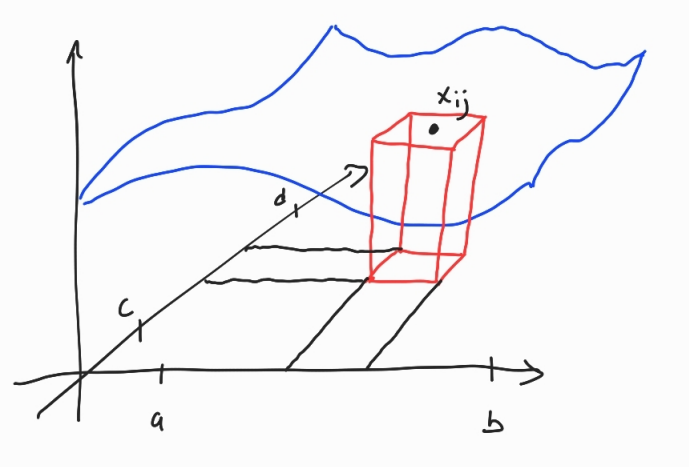


Figure 2: One rectangular prism in the Riemann sum for the blue surface.

This gives us an approximation, known as a **double Riemann sum**, for the volume under the surface, given by

$$\text{Volume} \approx \sum_{i=1}^{n}\sum_{j=1}^{m} g(x_i, y_j)\Delta A$$

Just as the single variable case, we can define the integral of g on the rectangle $[a, b] \times [c, d]$ by setting

$$\iint g(x, y)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} g(x_i, y_j)\Delta A$$

when the limit exists. If the limit exists, we say that g is **integrable** on the region $[a, b]\times[c, d]$. Most functions that you are familiar with, in particular continuous functions, will be integrable.

Just as before, this definition takes into account the sign of the volume, as if $g(x_i, y_j) < 0$, the corresponding rectangular prism will have negative volume. However, if $g(x, y) > 0$ on the entire region $[a, b] \times [c, d]$, then the volume under the curve is given by the double integral.

Almost all of the integration rules from single-variable calculus still work: the integral of a sum is the sum of the integrals, constants can come out of the integral, etc.

While this limit definition works great as a definition, in practice it isn't very useful to help us compute things. Let's suppose we wanted to integrate a function $f(x, y)$ on $[a, b] \times [c, d]$. If we fix a specific value of x, then the integral

$$\int_c^d f(x, y)\, dy$$

gives a number which depends on x. We could then write down a function

$$A(x) = \int_c^d f(x, y)\, dy$$

which takes an x value as input and returns the value of the corresponding integral in y. Geometrically for some fixed x, $A(x)$ should correspond to the cross-sectional area of the region corresponding to the given x value.

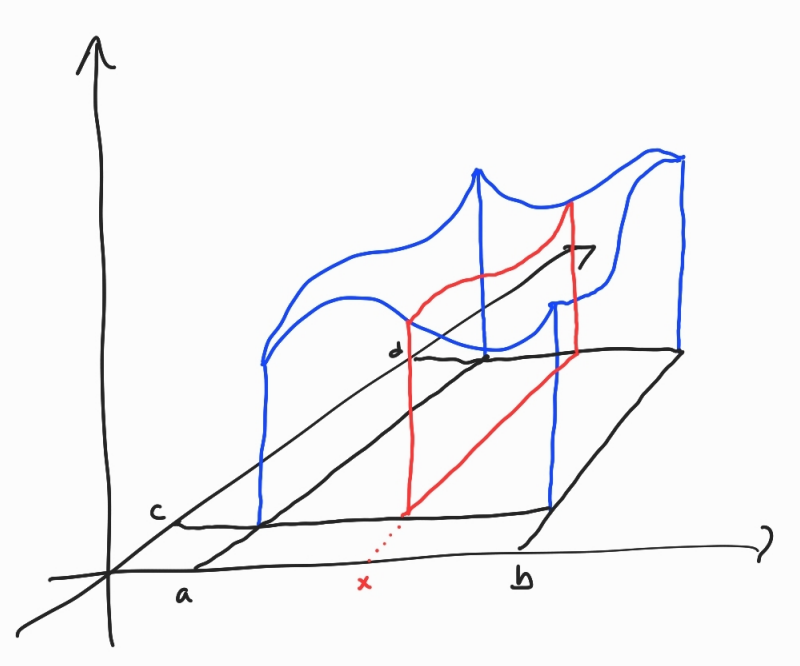


Figure 3: The cross-sectional area corresponding to $A(x)$ is outlined in red.

We could then integrate over each choice of $x \in [a, b]$ to obtain our total volume. Thus, we can write

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

This is called an **iterated integral**, and we'll usually omit the parentheses and simply write

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

and we compute the integral from the inside out.

Example. Suppose we want to compute

$$\iint_{[0,3]\times[1,2]} x^2 y\,dA.$$

We can then write

$$\begin{aligned}
\iint_{[0,3]\times[1,2]} x^2 y\,dA &= \int_0^3 \int_1^2 x^2 y\,dy\,dx \\
&= \int_0^3 \left(\frac{1}{2} x^2 y^2 \Big|_1^2 \right) dx \\
&= \int_0^3 \left(\frac{4}{2} x^2 - \frac{1}{2} x^2 \right) dx \\
&= \int_0^3 \frac{3}{2} x^2\,dx \\
&= \frac{3}{2} \int_0^3 x^2\,dx
\end{aligned}$$

$$= \frac{3}{2}\left(\frac{1}{3}x^3\Big|_0^3\right)$$
$$= \frac{3}{2}(9-0)$$
$$= \boxed{\frac{27}{2}}$$

We don't have to integrate in y first, so just for fun, let's try integrating it the other way.

$$\begin{aligned}\iint_{[0,3]\times[1,2]} x^2 y\,dA &= \int_1^2 \int_0^3 x^2 y\,dx\,dy \\ &= \int_1^2 \left(\frac{1}{3}x^3 y\Big|_0^3\right) dy \\ &= \int_1^2 9y\,dy \\ &= \frac{9}{2}y^2\Big|_1^2 \\ &= \frac{36}{2} - \frac{9}{2} \\ &= \boxed{\frac{27}{2}}\end{aligned}$$

It turns out that these values agreeing is not a coincidence - this result is known as **Fubini's Theorem**. Colloquially, if we integrate a function on a rectangle, the order of integration does not matter. An interesting consequence of this is if we are integrating a function

$$f(x,y) = g(x)h(y)$$

which is the product of a function only in x and a function only in y. We then have that

$$\begin{aligned}\iint_{[a,b]\times[c,d]} g(x)h(y)\,dA &= \int_a^b \left(\int_c^d g(x)h(y)\,dy\right) dx \\ &= \int_a^b g(x)\left(\int_c^d h(y)\,dy\right) dx \qquad \text{(Since } g(x) \text{ is constant in } y\text{)} \\ &= \left(\int_c^d h(y)\,dy\right)\left(\int_a^b g(x)\,dx\right) \qquad \text{(Since } \int_c^d h(y)\,dy \text{ is constant in } x\text{)}\end{aligned}$$

and so it suffices to compute the two integrals separately, and take their product. This is a very special case! We should not expect to be able to split up one double integral into two single integrals in general.

A Note on Notation: I will sometimes move my differentials to come immediately after to the integral, rather than at the end of the integrand, as to more easily keep track of which integral belongs to which variable. For example, I will sometimes write

$$\int_a^b dx \int_c^d dy\, f(x,y)$$

to denote the integral

$$\int_a^b \int_c^d f(x,y)\,dy\,dx.$$

This notation is not very common, and you are free to use whichever notation you like - just be aware of what I mean when I write this.

You may also recall from single-variable calculus that there exists a formula to compute the average value of a function $f(x)$ on an interval $[a, b]$, given by

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

We also have a two dimensional analog for a function $g(x, y)$, where we replace the denominator with the area of the region we are integrating over, to obtain the formula

$$g_{\text{ave}} = \frac{1}{(b-a)(d-c)} \iint_{[a,b]\times[c,d]} g(x, y)\, dA$$

for the average value of g on this region.

You should think about double integrals in the following way: we divide our region into tiny pieces, and have some quantity assigned to that region. Then, integrating over the region gives us a sum of the assigned quantity over the region. This quantity could be the constant function 1, in which case we are computing area, or some function, in which case we are computing the volume under the surface it defines.

2 Double Integrals over General Regions

Textbook Reference: Chapter 15.2.
Planned Lecture Date(s): March 29, 2023.

Unfortunately, most double integrals that occur in nature won't be over a rectangle - they'll probably live over some more complicated region.

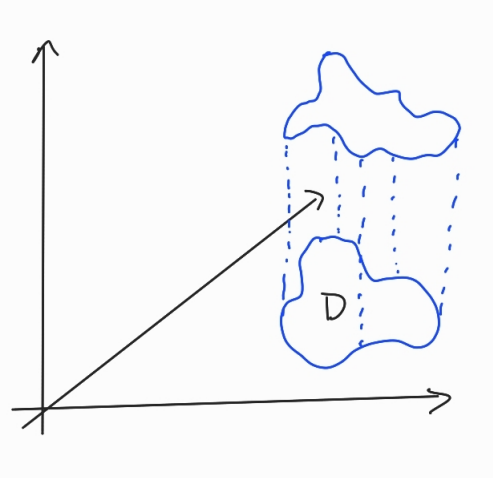


Figure 4: The region D is the domain of integration of the blue function.

When this region is nice enough, we can still do some meaningful integration over it. The textbook defines two types of regions:

- **Type I Regions:** Regions which lie between the graphs of two continuous functions of x, i.e.

  $$D = \{(x, y) \,|\, a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

  We can integrate over D by computing

  $$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

- **Type II Regions:** Regions which lie between the graphs of two continuous functions of y, i.e.

  $$E = \{(x, y) \,|\, c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

  We can integrate over E by computing

  $$\iint_E f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

The idea here is that if our domain can be viewed as the region between two curves, we can use these curves as our bounds of integration.

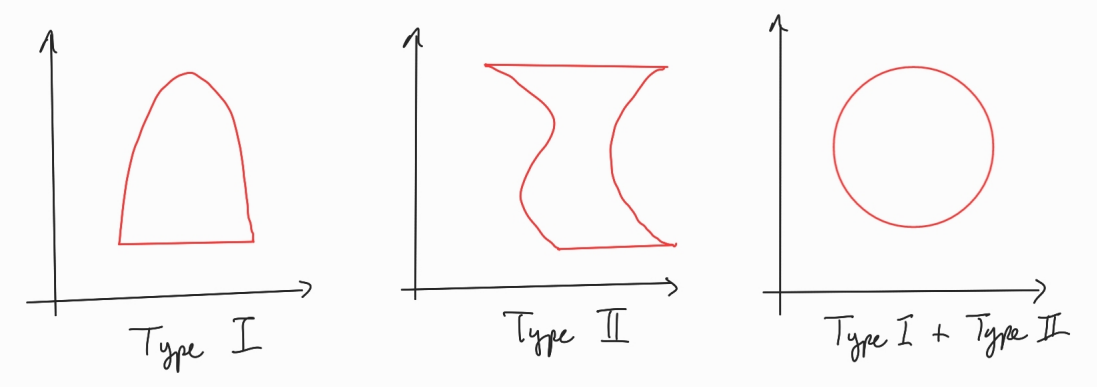


Figure 5: Regions of Type I and Type II.

The distinction between Type I and Type II is not that important: in practice, the best way to determine how to integrate over a region is to draw a picture and write down the bounds that make the most sense.

One application of integrating over general regions is if we want to determine the area of a region D, we can simply integrate the constant function 1 to obtain the desired area. Our formula is then given by

$$\text{Area}(D) = \iint_D 1 \, dA.$$

Let's try it with a problem that we already know how to solve.

Example. Find the area under the curve $y = x^2$ between $x = 0$ and $x = 4$.

We first draw a picture.

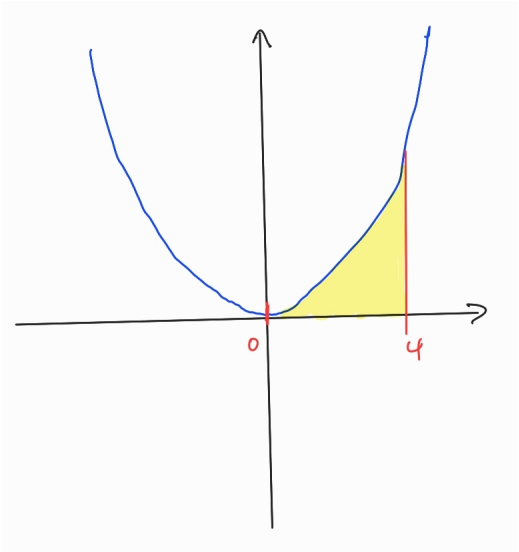


Figure 6: The region described in the problem.

From Calc II, we know that the area is given by

$$A = \int_0^4 x^2 \, dx = \frac{64}{3}.$$

Let's check this using our double integral. We can integrate it over x first, and then over y. The variable x ranges from 0 to 4, and at constant x, the variable y ranges from 0 to x^2. We therefore have

$$\begin{aligned}\iint_D 1\,dA &= \int_0^4 \int_0^{x^2} 1\,dy\,dx \\ &= \int_0^4 x^2\,dx \\ &= \boxed{\frac{64}{3}}\end{aligned}$$

which not only agrees with what we computed above, but the computation is also incredibly similar. We could however also integrate with y first. Since the point of intersection is $(4, 16)$, we have that y ranges from 0 to 16, and at constant y, x ranges from $\sqrt{y}$ (the blue curve, since $y = x^2$) to 4. We therefore have

$$\begin{aligned}\iint_D 1\,dA &= \int_0^{16} \int_{\sqrt{y}}^4 1\,dx\,dy \\ &= \int_0^{16} 4 - \sqrt{y}\,dy \\ &= 4y - \frac{2}{3}y^{\frac{3}{2}}\Big|_0^{16} \\ &= 64 - \frac{2}{3}(64) \\ &= \boxed{\frac{64}{3}}\end{aligned}$$

which matches our previous result.

It's good to be able to parameterize regions which are both Type I and Type II in two different ways, as sometimes one way will give an easier integral than the other.

Example. Evaluate $\iint_D x + 2y\,dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

The first thing we should do is draw a picture of the region.

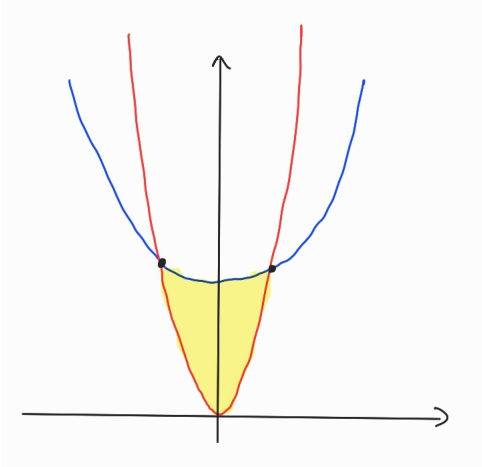


Figure 7: The region D shaded in yellow. The curve $y = 1 + x^2$ is in blue and $y = 2x^2$ is in red.

Since the region is bounded by two continuous functions of x, we will use those functions as our bounds of integration (this would be a Type I region). We first identify that the two points of intersection are $(\pm 1, 2)$.

From there, looking at our picture, we see that x will go from -1 to 1. At a constant value of x, y will go from the red curve to the blue curve, or in other words, from $2x^2$ to $1+x^2$. We then set up our integral as

$$\iint_D x + 2y \, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} x + 2y \, dy \, dx$$

From here, we evaluate our integral, being careful to plug in the variables in the bounds correctly.

$$\begin{aligned}
\int_{-1}^{1} \int_{2x^2}^{1+x^2} x + 2y \, dy \, dx &= \int_{-1}^{1} \left(xy + y^2 \Big|_{2x^2}^{1+x^2} \right) dx \\
&= \int_{-1}^{1} x(1+x^2) + (1+x^2)^2 - 2x^3 - 4x^4 \, dx \\
&= \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1 \, dx \\
&= -3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x \Big|_{-1}^{1} \\
&= \boxed{\frac{32}{15}}
\end{aligned}$$

This computation was a little bit tedious. Perhaps we can save some time with a few integration tricks! Let's think about what happens when we integrate over a region E which is symmetric across the y-axis.

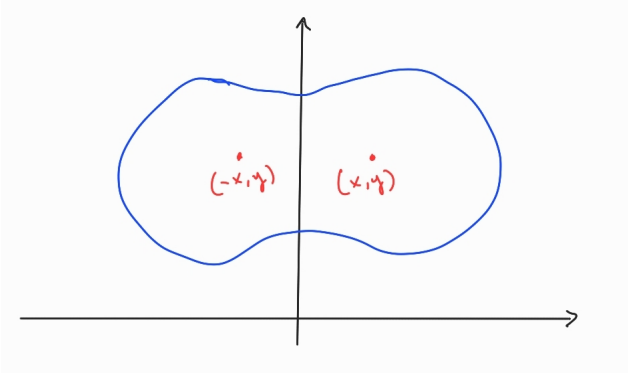


Figure 8: The blue region E is symmetric in the x coordinate.

For each point (x, y) in E, there is a corresponding point $(-x, y)$ in E. If the function we are integrating is an odd function in x (in other words, $f(-x, y) = -f(x, y)$), then whatever value f takes at (x, y), it will take the negative of that value at $(-x, y)$. Thus, those two points together will contribute a total of 0 towards the value of the integral. Since we can pair up all of the points in the region in this manner, as E is symmetric, it turns out that the value of $\iint_E f(x, y) \, dA$ is actually zero! The point to remember is that odd functions integrated over a symmetric region are zero - before integrating, one should always check to see if things will cancel by symmetry.

Armed with this knowledge, lets return to our example: the region bounded by $y = 1 + x^2$ and $y = 2x^2$ is symmetric in x, and the function x is certainly an odd function of x. Thus, we have that

$$\iint_D x + 2y \, dA = \iint_D x \, dA + \iint_D 2y \, dA = 0 + \iint_D 2y = \iint_D 2y$$

which could save us a fair amount of work while integrating! Unfortunately, since $2y$ is not an odd function of x, we cannot apply the same trick (The function $2y$ is an odd function in y, but the region D is not

symmetric in y). Odd functions occur all the time in nature - most notably, polynomials with only odd powers of x are odd, as is the sine and tangent functions (think about their Taylor series!).

You can apply a similar trick to even functions ($f(-x, y) = f(x, y)$), but it isn't quite as useful: since the value at one point in the pair is the same as the other, we can simply integrate over half the region and double the result we get.

Example. Let D be the region bounded by $x = 4 - y^2$ and $x = y^2 - 4$. Compute $\iint_D x^3y^2 + xy + 3\,dA$.

As before, we first draw our picture.

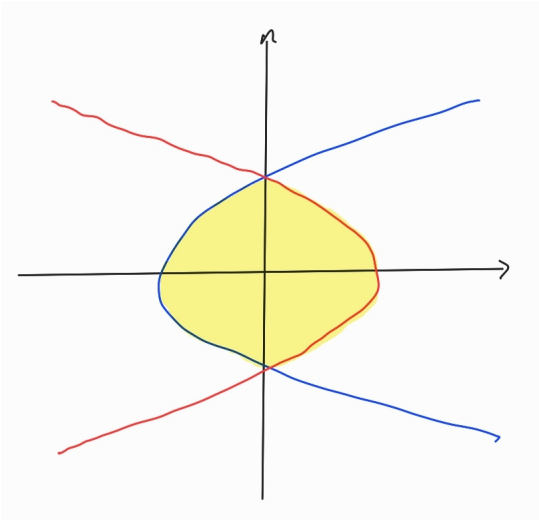


Figure 9: The region D is shaded in yellow. The blue curve is $x = y^2 - 4$, and the red curve is $x = 4 - y^2$.

Since the region is bounded by two functions of y, we will integrate over y first, then x as a function of y (this is a Type II region). The points of intersection are $(0, \pm 2)$, so y will range from -2 to 2. At constant y, x will range from the blue curve (smaller x) to the red curve (larger x). Thus, we set up our integral as follows.

$$\iint_D x^3y^2 + xy + 3\,dA = \int_{-2}^{2}\int_{y^2-4}^{4-y^2} x^3y^2 + xy + 3\,dx\,dy$$

At this point, we check for symmetry cancellations. Since the region is symmetric in both x and y, any odd function of x or y will integrate to zero. We notice that x^3y^2 is odd in x, and xy is odd in both x and y, so the only remaining term to integrate is 3. We then compute

$$\begin{aligned}
\int_{-2}^{2}\int_{y^2-4}^{4-y^2} 3\,dx\,dy &= \int_{-2}^{2}\left(3x\Big|_{y^2-4}^{4-y^2}\right)dy \\
&= 3\int_{-2}^{2}(4-y^2)-(y^2-4)\,dy \\
&= 3\int_{-2}^{2} 8-2y^2\,dy \\
&= 6\int_{0}^{2} 8-2y^2\,dy \\
&= 6\left(16-\frac{16}{3}\right) \\
&= \boxed{64}
\end{aligned}$$

It turns out that we could have computed the area of this region by integrating over x first, if we had wanted to instead - the work would just be slightly messier. The values of x range from -4 to 4, but the bounds on y are no longer one single function of x - they change at $x = 0$. This is not an issue, since all we have to do is split up the domain into two regions and add them! For the blue curve, we have that $x = y^2 - 4$, so $y = \pm\sqrt{4+x}$, and for the red curve, we have that $x = 4 - y^2$, so $y = \pm\sqrt{4-x}$. We then have

$$\begin{aligned}
\iint_D x^3y^2 + xy + 3\,dA &= \iint_D 3\,dA \\
&= \int_{-4}^{0}\int_{-\sqrt{4+x}}^{\sqrt{4+x}} 3\,dy\,dx + \int_{0}^{4}\int_{-\sqrt{4-x}}^{\sqrt{4-x}} 3\,dy\,dx \\
&= 3\int_{-4}^{0} 2\sqrt{4+x}\,dx + 3\int_{0}^{4} 2\sqrt{4-x}\,dx \\
&= 6\left(\frac{2}{3}(4+x)^{\frac{3}{2}}\Big|_{-4}^{0}\right) + 6\left(\frac{2}{3}(4-x)^{\frac{3}{2}}\Big|_{0}^{4}\right) \\
&= 6\left(\frac{16}{3}\right) + 6\left(\frac{16}{3}\right) \\
&= \boxed{64}
\end{aligned}$$

One may notice that the two integrals ended up evaluating to the same value of 32 - this is not a coincidence, as they correspond to integrating the function $f(x, y) = 3$, a symmetric function in x, over the two halves of the domain D, which is symmetric in x.

The moral of the story is that being able to set up our integral in two different ways will allow you to compute the easier of the two integrals.

3 Double Integrals in Polar Coordinates

Textbook Reference: Chapter 15.3.
Planned Lecture Date(s): March 31, 2023.

Thus far, all of our double integrals have been set up as integrals over dx and dy. This is convenient for the Type I and Type II regions we have encountered, but perhaps less so for other regions.

Suppose that $f(x, y)$ is a function on D, the unit disk, which is given by

$$D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}.$$

Then, we could certainly integrate f over D using the parameterization

$$\iint_D f \, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) \, dy \, dx.$$

This however does not feel very natural, as the bounds of integration are a bit messy, and philosophically, the x and y coordinates don't respect the natural symmetry of the circle. The region would be a lot easier to describe in polar coordinates! If we let

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

then we can write

$$D = \{(r, \theta) \,|\, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$$

and so D is essentially a polar "rectangle". We'll have to be a bit careful when we integrate though - let's try computing the area of the circle. We have that

$$\int_0^1 \int_0^{2\pi} 1 \, d\theta \, dr = \int_0^1 2\pi \, dr = 2\pi$$

which is concerning, because the area of a circle should be π! The reason for this is somewhat subtle. Let's take a look at what dA means.

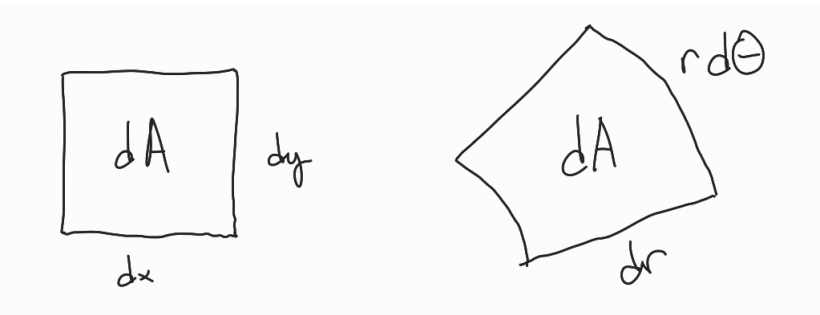


Figure 10: The infinitesimal area of dA computed in rectangular and polar coordinates.

Intuitively, we want to think of dA as a small chunk of area, which we divide our domain of integration into, and sum the value of our integrand over each chunk. In rectangular coordinates, a small chunk of area is a small rectangle, with width dx and height dy, corresponding to a small change in x and a small change in y, so we can confidently write

$$dA = \, dx \, dy$$

for the area of this small rectangle. In polar coordinates, things are a bit trickier. One side of the "rectangle" is still dr, a small change in r, but the other side is not $d\theta$! The other side of the "rectangle" is given by an arc of the circle of radius r, corresponding to an angle of $d\theta$, a small change in θ. Since the length of an arc

of a circle of radius L corresponding to an angle of φ is given by $L\varphi$, the length of the arc in our diagram is $r\,d\theta$. Since we are thinking of dr and $d\theta$ as very small, we can pretend that the arc is really a straight line for purposes of calculating area. Thus, we have a rectangle of width dr and length $r\,d\theta$, so we should have that

$$dA = r\,dr\,d\theta.$$

Let's see if this works! We compute the area of the circle as follows.

$$\begin{aligned}\iint_D 1\,dA &= \int_0^1 \int_0^{2\pi} r\,d\theta\,dr \\ &= 2\pi \int_0^1 r\,dr \\ &= 2\pi\left(\frac{1}{2}\right) \\ &= \pi\end{aligned}$$

which is what we were hoping for. This is good news! We can do this in general using the formula

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$$

where we make sure to not forget the r. This is the first example of an instance where we change our variables to make our computations easier, at the cost of some factor (called the **Jacobian**) in the integrand; this will reappear more prominently later on.

Another sign that perhaps something is not quite right with writing just $dr\,d\theta$ is that this should represent an area. Although r has dimensions of length, θ is dimensionless (since it is the ratio of two lengths), so the quantity $dr\,d\theta$ has dimensions of length, whereas an area should have dimensions of length^2 (which the factor of r indeed fixes).

Example. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

The first thing we always do is draw a picture.

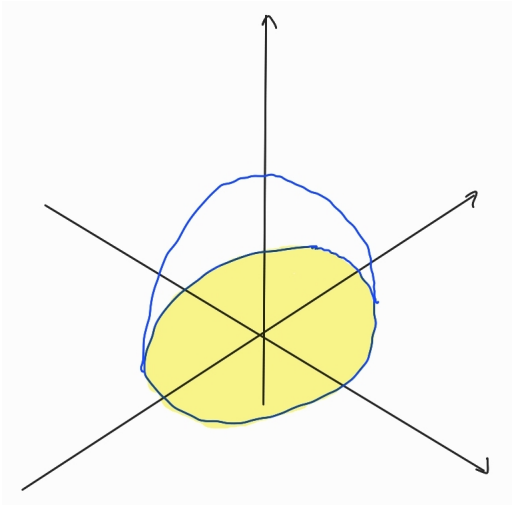


Figure 11: The blue surface is the paraboloid, and the yellow shaded region is the domain of integration.

From the picture (or computing algebraically), we can see that the paraboloid intersects $z = 0$ along the circle $x^2 + y^2 = 1$, so we should let this disk, call it D, be our region of integration. At each point (x, y),

we'd like to sum the value of $z = 1 - x^2 - y^2$. Thus, we have

$$\begin{aligned}
\iint_D z\,dA &= \iint_D 1 - x^2 - y^2\,dA \\
&= \int_0^1 \int_0^{2\pi} \left(1 - r^2\cos^2(\theta) - r^2\sin^2(\theta)\right) r\,d\theta\,dr \\
&= \int_0^1 \int_0^{2\pi} r(1 - r^2)\,d\theta\,dr \\
&= 2\pi \int_0^1 r - r^3\,dr \\
&= 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) \\
&= \boxed{\frac{\pi}{2}}
\end{aligned}$$

which gives us the volume of the region. You can try doing a similar calculation using rectangular coordinates, but I would imagine that it would not be a very fun calculation.

Example. Find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.

Let's draw a picture of the region in question.

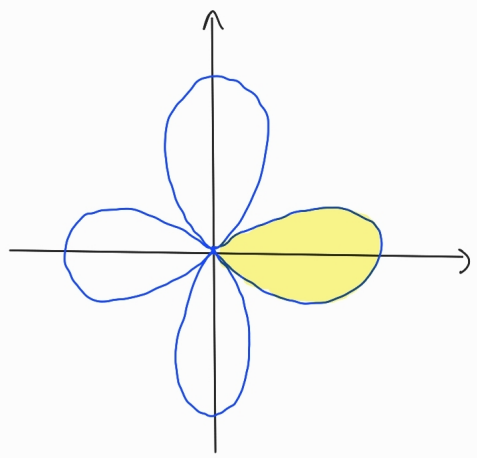


Figure 12: The curve $r = \cos(2\theta)$ in blue.

The "petals" of the rose are bounded by when $r = 0$, which occurs when $\cos(2\theta) = 0$, and thus when $\theta = \pm\frac{\pi}{4}$. At any given value of θ, r will range from 0 to the blue curve, which is $r = \cos(2\theta)$. We then integrate as follows.

$$\begin{aligned}
\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos(2\theta)} r\,dr\,d\theta &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\cos^2(2\theta)\,d\theta \\
&= \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(4\theta) + 1\,d\theta \\
&= \frac{1}{4}\left(\frac{1}{4}\sin(4\theta) + \theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\right) \\
&= \frac{1}{4}\left(\frac{\pi}{4} + \frac{\pi}{4}\right)
\end{aligned}$$

$$= \boxed{\frac{\pi}{8}}$$

Computing the area of this region using rectangular coordinates would be a nightmare, but using polar coordinates, it's extremely doable! This is the benefit of changing variables, and we'll see more examples of this later on.

4 Applications of Double Integrals

Textbook Reference: Chapter 15.4.
Planned Lecture Date(s): April 3, 2023.

Intuitively, we want to think of integration as the calculus version of breaking up a region into little chunks, giving some value to each chunk, and summing the values together to compute the total value for the entire region. One application of this is computing the mass of objects with variable density.

You should think of density as a function which, at every point in your region, tells you the mass per volume in a small chunk around that point. Thus, to compute mass, we should sum up the density at each chunk, multiplied by the length/area/volume of that chunk.

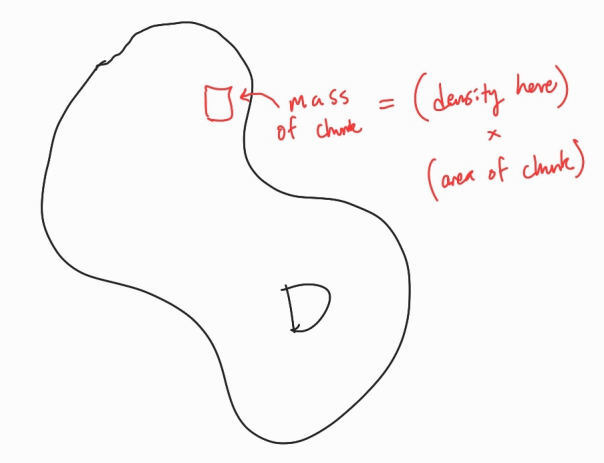


Figure 13: Total mass is given by integrating density everywhere.

A Note on Notation: In physics, you'll sometimes see one-dimensional density denoted by λ, two-dimensional density denoted by σ, and three-dimensional (and higher) density denoted by ρ.

Therefore, to compute the mass of a two-dimensional object with variable density, we should compute

$$\iint_D \sigma(x,y)\,dA.$$

Note in particular that if the density is constant, we could factor it out and simply integrate 1 over our region to get the area, and multiply by the constant density to get the mass. This should agree with the formula you are familiar with in physics, where

$$m = \sigma A$$

where m denotes mass and A denotes the area. Note that this equation is really a consequence of the statement that $dm = \sigma(x,y)\,dA$, which says that a small change in mass is equal to a small change in area times the density at that area. As such, we can integrate to find

$$m = \iint_D dm = \iint_D \sigma(x,y)\,dA$$

which is the formula we have above.

Example. Find the mass of a circular disk of radius 1, where the density is given by $\sigma(x,y) = x^2y^2$.

As always, we first draw a picture.

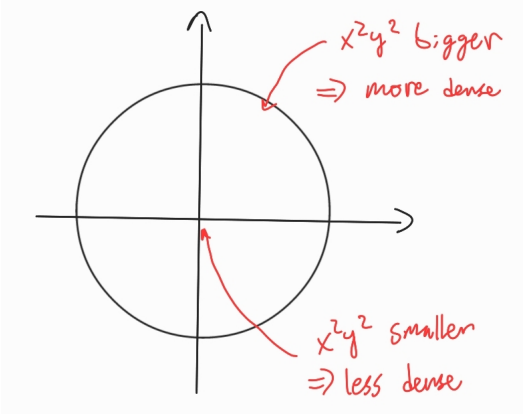


Figure 14: The region in question and where it is more/less dense.

We then have that the mass is given by our formula

$$\begin{aligned}
m &= \iint_D x^2y^2\,dA \\
&= \int_0^{2\pi}\int_0^1 r^5\cos^2(\theta)\sin^2(\theta)\,dr\,d\theta \\
&= \left(\int_0^1 r^5\,dr\right)\left(\int_0^{2\pi}\cos^2(\theta)\sin^2(\theta)\right) \\
&= \frac{1}{6}\left(\int_0^{2\pi}\frac{1}{4}\sin^2(2\theta)\,d\theta\right) \\
&= \frac{1}{24}\left(\int_0^{2\pi}\frac{1-\cos(4\theta)}{2}\,d\theta\right) \\
&= \frac{1}{24}\left(\frac{1}{2}\theta - \frac{1}{8}\sin(4\theta)\Big|_0^{2\pi}\right) \\
&= \boxed{\frac{\pi}{24}}
\end{aligned}$$

Another application of double integrals is computing moments and centers of mass. Given a region D with density $\sigma(x,y)$, we can compute the **moment about the x-axis** via the formula

$$M_x = \iint_D y\sigma(x,y)\,dA$$

Note that the variable in the integrand is y. We can analogously define the **moment about the y-axis** by computing

$$M_y = \iint_D x\sigma(x,y)\,dA$$

These quantities have physical significance primarily in computing centers of mass, for which we have the formula

$$(\overline{x},\overline{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right) = \left(\frac{\iint_D x\sigma(x,y)\,dA}{\iint_D \sigma(x,y)\,dA},\frac{\iint_D y\sigma(x,y)\,dA}{\iint_D \sigma(x,y)\,dA}\right)$$

where $(\overline{x},\overline{y})$ denotes the coordinates of the center of mass. Intuitively, we want to think about the center of mass as the point around which the mass of the region is evenly distributed, and so it represents an "average" location of the mass. This is represented in our formula, since (without loss of generality) the x-coordinate of the center of mass is given by a weighted average value for x, where the weighting is given by σ.

Example. Find the center of mass of a triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,2)$, with density $\sigma(x,y) = 1 + 3x + y$.

Note: The word **lamina** in the textbook refers to a planar (two-dimensional) region.

We draw our picture.

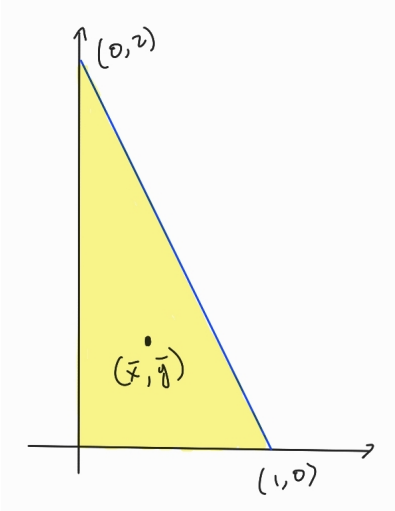


Figure 15: The triangular lamina.

We first identify that the bounding lines are the x-axis, the y-axis, and the line $y = -2x + 2$. To compute the center of mass, we need to compute three integrals: the moment about the x-axis, the moment about the y-axis, and the mass. We first compute the mass.

$$\begin{aligned}
m &= \iint_D dm \\
&= \iint_D \sigma(x,y)\, dA \\
&= \int_0^1 \int_0^{-2x+2} 1 + 3x + y \, dy \, dx \\
&= \int_0^1 (1+3x)(-2x+2) + \frac{1}{2}(-2x+2)^2 \, dx \\
&= \int_0^1 (-2x+2)(2+2x) \, dx \\
&= 4\int_0^1 1 - x^2 \, dx \\
&= \boxed{\frac{8}{3}}
\end{aligned}$$

We now compute the x-moment.

$$\begin{aligned}
M_y &= \iint_D x \, dm \\
&= \iint_D x\sigma(x,y)\, dA \\
&= \int_0^1 \int_0^{-2x+2} x + 3x^2 + xy \, dy \, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (x+3x^2)(-2x+2) + \frac{1}{2}x(-2x+2)^2\,dx \\
&= \int_0^1 (-2x+2)(2x+2x^2)\,dx \\
&= 4\int_0^1 x - x^3\,dx \\
&= \boxed{1}
\end{aligned}
$$

We similarly compute the y-moment.

$$
\begin{aligned}
M_x &= \iint_D y\,dm \\
&= \iint_D y\sigma(x,y)\,dA \\
&= \int_0^1 \int_0^{-2x+2} y + 3xy + y^2\,dy\,dx \\
&= \int_0^1 \frac{1}{2}(1+3x)(-2x+2)^2 + \frac{1}{3}(-2x+2)^3\,dx \\
&= \int_0^1 (2+2x-10x^2+6x^3) + \left(\frac{8}{3} - 8x + 8x^2 - \frac{8}{3}x^3\right)\,dx \\
&= \int_0^1 \left(\frac{14}{3} - 6x - 2x^2 + \frac{10}{3}x^3\right)\,dx \\
&= \left(\frac{14}{3} - \frac{6}{2} - \frac{2}{3} + \frac{10}{3}\cdot\frac{1}{4}\right) \\
&= \left(\frac{56-36-8+10}{12}\right) \\
&= \boxed{\frac{11}{6}}
\end{aligned}
$$

We thus conclude that

$$(\overline{x},\overline{y}) = \left(\frac{1}{\frac{8}{3}}, \frac{\frac{11}{6}}{\frac{8}{3}}\right) = \boxed{\left(\frac{3}{8}, \frac{11}{16}\right)}$$

which gives our desired center of mass.

Keep in mind our symmetry tricks here: if you're given a region and asked to find the center of mass, and the density is symmetric over one of the axes, then the density will be even as a function of that variable, and so multiplying by that variable to compute a moment will give 0. This matches our physical intuition that if an object has symmetric density, the center of mass should be on the axis of symmetry.

The last application of double integrals that I'll discuss are moments of inertia. In physics, we have formulas such as $F = ma$, $p = mv$, and $E = \frac{1}{2}mv^2$ to relate physical quantities such as force, mass, acceleration, etc. However, these formulas all describe linear motion - to describe rotational motion, we'll have to replace many of these quantities with analogous quantities. For example, linear velocity (denoted v) is replaced by angular velocity (denoted ω), force (usually F) is replaced with torque (usually τ), and mass (usually m) is replaced by the moment of inertia (usually I). Keep in mind that when we convert to rotational motion, all motion, and therefore all relevant quantities, are computed relative to an axis of rotation.

For a point mass (a small object where all mass is concentrated at one point), the formula for moment of inertia is given by

$$I_{\text{point}} = mr^2$$

where m denotes the mass of the point mass, and r denotes the distance from the axis of rotation. For larger objects, we can use our knowledge of calculus to compute the moment of inertia for each chunk of our region, and integrate to find the total moment of inertia. Thus, our formula for moment of inertia is given by

$$I = \iint_D r_\perp^2 \, dm = \iint_D r_\perp^2 \sigma(x, y) \, dA$$

where $r_\perp$ is the distance from a point in D to the axis of rotation (not to be confused with the r from polar coordinates).

Example. Let D be the annulus of inner radius R_1 and outer radius R_2, centered at the origin, with constant density σ. Compute the moment of inertia of D, as a function of R_1 and R_2, rotated around the z-axis.

Note: An **annulus** is the region between two concentric circles in the plane. Informally, you should think of it as a two-dimensional donut.

As usual, we first draw a picture of the region.

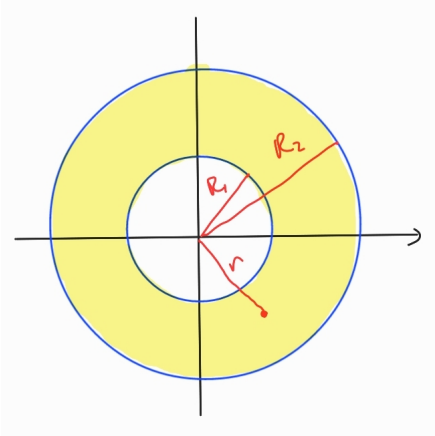


Figure 16: The annular region described in the problem.

We compute our moment of inertia using the formula. Since we are rotating around the z-axis, the distance from any point (x, y) to the axis of rotation is $\sqrt{x^2 + y^2}$. This is more easily represented in polar coordinates, where $r_\perp = r$.

$$\begin{aligned} I &= \iint_D r_\perp^2 \, dM \\ &= \iint_D r_\perp^2 \sigma \, dA \\ &= \sigma \int_0^{2\pi} \int_{R_1}^{R_2} r^3 \, dr \, d\theta \\ &= \boxed{2\pi\sigma \left(\frac{1}{4}(R_2^4 - R_1^4) \right)} \end{aligned}$$

Note additionally that since the density is constant, the mass of D is given by

$$m = \iint_D \sigma \, dA = \sigma \iint_D dA = \sigma \cdot \pi(R_2^2 - R_1^2)$$

Thus, if we rewrite our expression for the moment of inertia using this expression for mass, we have that

$$I = 2\pi\sigma \left(\frac{1}{4}(R_2^4 - R_1^4) \right)$$

$$= \frac{1}{2}\pi\sigma(R_2^2 + R_1^2)(R_2^2 - R_1^2)$$
$$= \frac{1}{2}m(R_2^2 + R_1^2)$$

which gives the moment of inertia as a function of the mass, rather than the density. Setting $R_1 = 0$, which makes the region D into a disk of radius R_2, we get that

$$I_{\text{disk}} = \frac{1}{2}mR_2^2.$$

Setting $R_1 = R_2$, which makes D into a "ring" of radius R_2 around the z-axis, we get that

$$I_{\text{ring}} = mR_2^2.$$

Students familiar with moments of inertia from physics may find these formulas familiar (but this is how they are derived!).

In the special case of computing the moment of inertia of a planar region, for any point (x, y) in our region, we have that the distance to the x-axis is given by y, the distance to the y-axis is given by x, and the distance to the z-axis is given by $\sqrt{x^2 + y^2}$. This has the interesting consequence that

$$I_z = \iint_D x^2 + y^2 \, dm = \iint_D x^2 \, dm + \iint_D y^2 \, dm = I_y + I_x$$

which physics students may recognize as the **perpendicular axis theorem**.

5 Surface Area

Textbook Reference: Chapter 15.5.
Planned Lecture Date(s): April 5, 2023.

Let's start this discussion with the area of a parallelogram. Given vectors $\vec{u}$ and $\vec{v}$ in three dimensional space, we can find the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ by computing the cross product of the vectors.

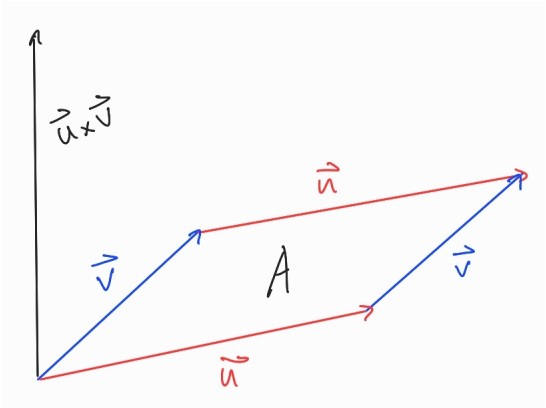


Figure 17: Area of a parallelogram spanned by $\vec{u}$ and $\vec{v}$.

The area A is then given by the magnitude of the cross product, or in other words, $A = |\vec{u} \times \vec{v}|$. A reminder that the cross product of two vectors

$$\vec{u} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}} \qquad \vec{v} = d\hat{\mathbf{i}} + e\hat{\mathbf{j}} + f\hat{\mathbf{k}}$$

is defined by

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a & b & c \\ d & e & f \end{vmatrix} \\ &= (bf - ce)\hat{\mathbf{i}} + (cd - af)\hat{\mathbf{j}} + (ae - bd)\hat{\mathbf{k}}\end{aligned}$$

and this product is anti-symmetric (i.e., $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$). Geometrically, the cross product of two vectors gives a vector with magnitude $\|\vec{u}\| \, \|\vec{v}\| \sin(\theta)$, where θ is the angle between $\vec{u}$ and $\vec{v}$, and is orthogonal to $\vec{u}$ and $\vec{v}$. This description, however, doesn't uniquely determine the cross product, since for any vector $\vec{w}$ which satisfies these conditions, the vector $-\vec{w}$ also works (in three dimensions, it turns out this is the only ambiguity that needs to be resolved). The common convention taken is the **right-hand rule**, which is the mathematical statement that $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, rather than $-\hat{\mathbf{k}}$ (and this is encapsulated in the determinant formula above). The name comes from the mnemonic that if you, using your right hand, point in the $\vec{u}$ direction with your index finger, and in the $\vec{v}$ direction with your middle finger, then your thumb should point in the direction of $\vec{u} \times \vec{v}$.

Disclaimer: The Fundamental Theorem of the Right-Hand Rule states that there always exists a way to do the right-hand rule without hurting yourself. If you find yourself twisting your arm in a manner that is painful, there is probably a less uncomfortable way to represent the same vectors with your fingers.

For a function $f(x, y)$, recall from Math 126 that the partial derivatives of f give the instantaneous rate of change of f in the x direction (and analogously for y). Thus, the tangent plane best approximating f at a point (x_0, y_0) changes $f_x(x_0, y_0)$ in the z direction for each unit it changes in the x direction (and again analogously for y). Thus, the tangent directions at a point (x_0, y_0) are given by

$$\vec{T_1} = \vec{i} + f_x(x_0, y_0)\vec{k} \qquad \vec{T_2} = \vec{j} + f_y(x_0, y_0)\vec{k}$$

where f_x and f_y denote the partial derivatives of f with respect to x and y.

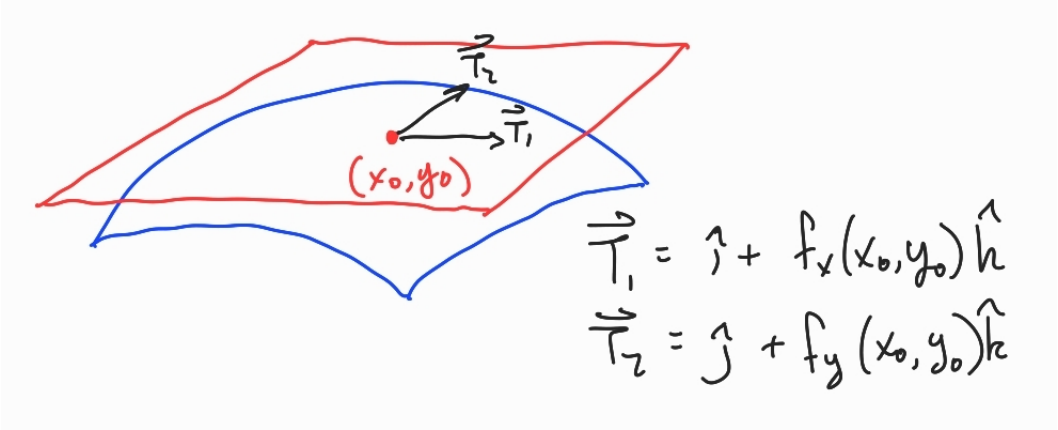


Figure 18: Tangent vectors to a surface.

Suppose we have a function $f(x, y)$ over a region D. We can then compute surface area of f using the following procedure. Divide D into small chunks, and for each chunk, we sum the area of the tangent plane above that chunk. Since the tangent plane approximates the surface above that chunk (and will do so better the smaller our chunk is), by summing the area of the tangent plane over all such chunks, we should obtain the total surface area.

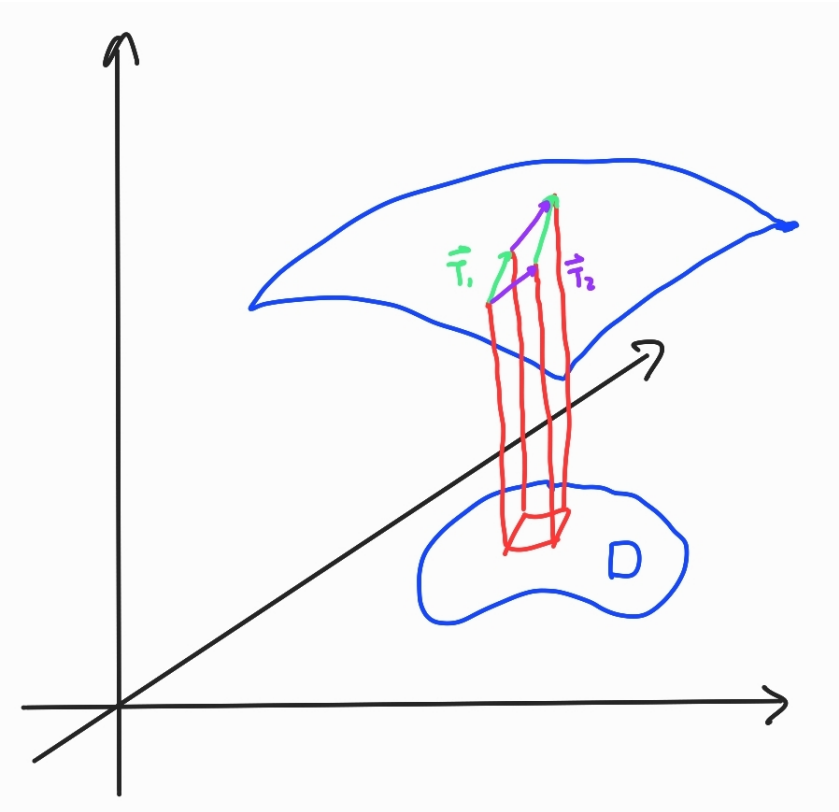


Figure 19: Breaking down our surface into small chunks, and approximating with tangent planes.

Mathematically, we should formalize this as follows. For each chunk dA, we should sum the area of the

tangent plane above dA. This is given by the area of the parallelogram spanned by $\vec{T_1}$ and $\vec{T_2}$. We thus have

$$\begin{aligned}\text{Area over } dA &= \left|\vec{T_1} \times \vec{T_2}\right| \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \\ &= \left|-f_x\hat{\mathbf{i}} - f_y\hat{\mathbf{j}} + \hat{\mathbf{k}}\right| \\ &= \sqrt{f_x^2 + f_y^2 + 1}\end{aligned}$$

Therefore, this is the quantity we should integrate over D. We have then that the surface area is given by

$$A(S) = \iint_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\, dA.$$

Example. Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

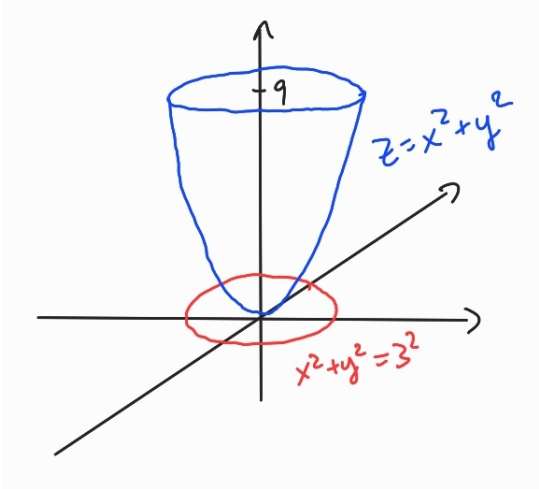


Figure 20: The paraboloid $z = x^2 + y^2$ over the disk $x^2 + y^2 = 9$.

Since the plane $z = 9$ cuts off the paraboloid along a disk of radius 3, we should use $D = \{x^2 + y^2 \leq 3^2, z = 0\}$ as our domain of integration (think of this as the "shadow" of our surface). Then, $f(x, y) = x^2 + y^2$. We compute

$$f_x = 2x \qquad f_y = 2y$$

and thus our desired integral is

$$\iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA.$$

We compute this using polar coordinates. We have

$$\begin{aligned}\iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA &= \int_0^{2\pi}\int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\ &= 2\pi \int_0^3 r\sqrt{1 + 4r^2}\, dr \\ &= \frac{\pi}{4}\int_1^{37} \sqrt{u}\, du \qquad (u = 1 + 4r^2)\end{aligned}$$

$$= \frac{\pi}{4}\left(\frac{2}{3}u^{\frac{3}{2}}\Big|_1^{37}\right)$$
$$= \boxed{\frac{\pi}{6}\left(37\sqrt{37}-1\right)}$$

which gives us our desired surface area.

Example. Find the surface area of a sphere with radius R.

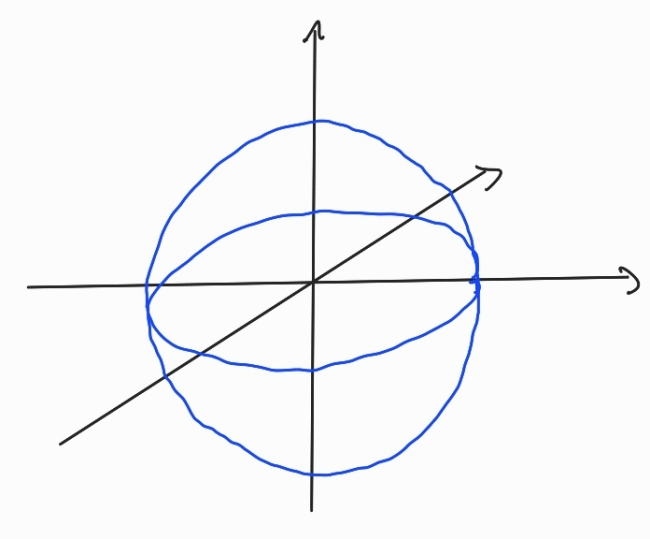


Figure 21: The sphere of radius R.

Since the sphere cannot be written as $z = f(x, y)$ (as it does not satisfy the vertical line test), we have to get creative. Let's integrate over the top hemisphere, and double our result. Since the sphere is given by $x^2 + y^2 + z^2 = R^2$, we can solve

$$z = \pm\sqrt{R^2 - x^2 - y^2}$$

and a choice of $+$ gives us the top hemisphere, whose "shadow" is $x^2 + y^2 \leq R^2$. Note then that

$$f_x = \frac{-x}{\sqrt{R^2 - x^2 - y^2}} \qquad f_y = \frac{-y}{\sqrt{R^2 - x^2 - y^2}}$$

We can then integrate

$$\begin{aligned}
\iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA &= \iint_D \sqrt{1 + \frac{x^2 + y^2}{R^2 - x^2 - y^2}}\, dA \\
&= \iint_D \sqrt{\frac{R^2}{R^2 - x^2 - y^2}}\, dA \\
&= \int_0^{2\pi}\int_0^R \sqrt{\frac{R^2}{R^2 - r^2}}\, r\, dr\, d\theta \\
&= 2\pi R \int_0^R r\sqrt{\frac{1}{R^2 - r^2}}\, dr \\
&= -\pi R \int_{R^2}^0 \sqrt{\frac{1}{u}} \qquad (u = R^2 - r^2)
\end{aligned}$$

$$
\begin{aligned}
&= \pi R \int_0^{R^2} u^{-\frac{1}{2}}\, du \\
&= \pi R \left(2\sqrt{u} \Big|_0^{R^2} \right) \\
&= 2\pi R^2
\end{aligned}
$$

to find the surface area of a hemisphere. We then double to find that the total surface area is $\boxed{4\pi R^2}$, which agrees with the formula we are familiar with.

6 Triple Integrals

Textbook Reference: Chapter 15.6.
Planned Lecture Date(s): April 7, 2023.

Just as with double integrals, we can define triple integrals in a similar way. Let's start with rectangles: divide up our rectangle $B = [a, b] \times [c, d] \times [e, f]$ into small boxes of size $dV = dx\, dy\, dz$, and define the integral to be

$$\iiint_B f(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\, dV.$$

Just as before, although this is a rigorous definition, it's not as convenient for computing integrals. We can more easily compute integrals using Fubini's Theorem, which tells us that

$$\iiint_B f(x, y, z)\, dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

and furthermore, we can integrate the variables in whichever order we want.

Just as in the double integral case, we should think of a triple integral in the following way: we divide up our region into tiny pieces, and sum some quantity assigned to each piece. The integral will then give us the sum of that quantity over the entire region of integration. In particular, if we'd like to compute volume, we can do so by integrating the constant function 1 over the region.

Of course, we'd want to integrate over more than just rectangles. The book again makes distinctions between the different types of regions over which we can integrate, but this is best illustrated again through example. We should find the bounds on each variable in our domain of integration, and use this to set up our integral.

Example. Find the volume of region in the first quadrant bounded by the coordinate planes, together with the plane $x + y + z = 1$.

Drawing a picture here is essential, as it's not easy to figure out the bounds we need.

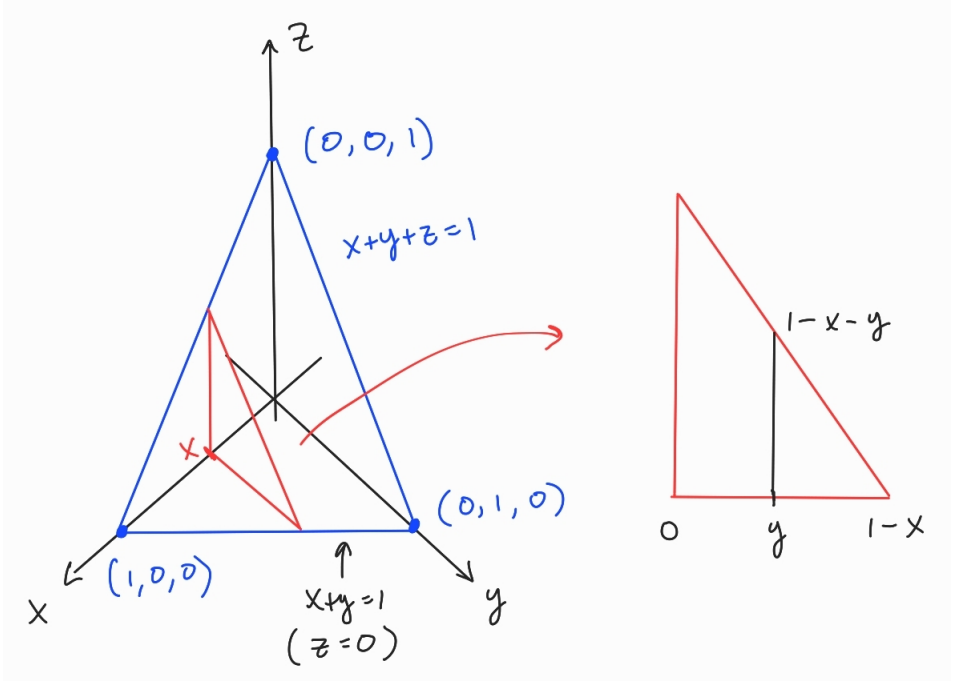


Figure 22: The region of integration.

We can really parameterize this in whichever order we want - let's do it alphabetically. The variable x ranges from 0 to 1, and at constant x, we see that y starts at 0 and goes to the line $x + y = 1$, since it lives in the plane $z = 0$. Thus, y ranges from 0 to $1 - x$. Finally, at constant x and y, the variable z ranges from 0 to the surface $x + y + z = 1$, and thus z goes from 0 to $1 - x - y$. We therefore set up our integral as follows.

$$\begin{aligned}
\iiint_E 1\,dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{1-x} 1 - x - y\,dy\,dx \\
&= \int_0^1 (1-x) - x(1-x) - \frac{1}{2}(1-x)^2\,dx \\
&= \int_0^1 \frac{1}{2} - x + \frac{1}{2}x^2\,dx \\
&= \frac{1}{2} - \frac{1}{2} + \frac{1}{6} \\
&= \boxed{\frac{1}{6}}
\end{aligned}$$

We can apply triple integrals in many of the same ways as double integrals. For example, we can compute centers of mass. In three variables, we define the moments about the coordinate planes to be

$$M_{yz} = \iiint_E x\,dm \qquad M_{xz} = \iiint_E y\,dm \qquad M_{xy} = \iiint_E z\,dm$$

where $dm = \rho(x, y, z)\,dV$. Similarly, the center of mass is given by

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{\iiint_E x\,dm}{m}, \frac{\iiint_E y\,dm}{m}, \frac{\iiint_E z\,dm}{m} \right).$$

Example. Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$, and $x = 1$.

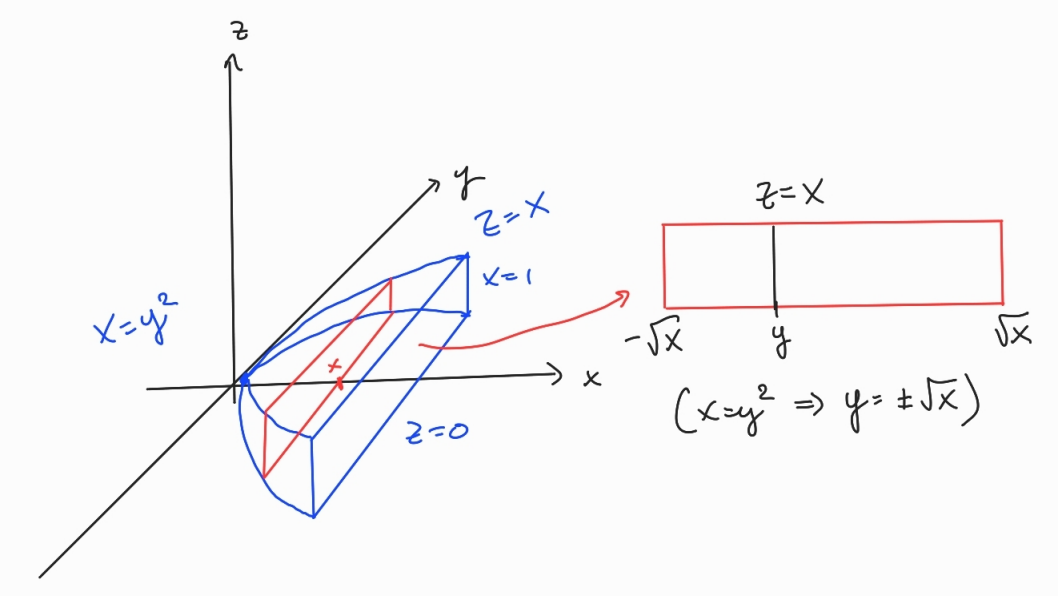


Figure 23: The slice of the parabolic cylinder.

We first find the mass. Let E denote the region of integration. From the diagram, we see that x goes from 0 to 1. At a constant x, y ranges from $-\sqrt{x}$ to $\sqrt{x}$. Finally, at constant x and y, z goes from 0 to x. Thus, we have the setup

$$\begin{aligned} m = \iiint_E dm &= \iint_E \rho \, dV \\ &= \rho \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^x 1 \, dz \, dy \, dx \\ &= \rho \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} x \, dy \, dx \\ &= \rho \int_0^1 2x^{\frac{3}{2}} \, dx \\ &= \frac{4}{5}\rho. \end{aligned}$$

We then find the centers of mass. Since the region (and the density) is symmetric in y, we know that $\overline{y} = 0$. We compute

$$\begin{aligned} \overline{x} &= \frac{\iiint_E x \, dm}{m} \\ &= \frac{5}{4\rho} \iiint_E x\rho \, dV \\ &= \frac{5}{4\rho}\rho \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^x x \, dz \, dy \, dx \\ &= \frac{5}{4} \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} x^2 \, dy \, dx \\ &= \frac{5}{4} \int_0^1 2x^{\frac{5}{2}} \, dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{5}{4} \cdot \frac{4}{7} \\
&= \frac{5}{7}.
\end{aligned}
$$

We similarly compute

$$
\begin{aligned}
\overline{z} &= \frac{\iiint_E z \, dm}{m} \\
&= \frac{5}{4\rho} \iiint_E z\rho \, dV \\
&= \frac{5}{4\rho}\rho \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^x z \, dz \, dy \, dx \\
&= \frac{5}{4} \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{2} x^2 \, dy \, dx \\
&= \frac{5}{4} \int_0^1 x^{\frac{5}{2}} \, dx \\
&= \frac{5}{4} \cdot \frac{2}{7} \\
&= \frac{5}{14}.
\end{aligned}
$$

We conclude that the center of mass is at $\boxed{\left(\frac{5}{7}, 0, \frac{5}{14}\right)}$.

7 Triple Integrals in Cylindrical Coordinates

Textbook Reference: Chapter 15.7.
Planned Lecture Date(s): April 10, 2023.

Just as in the double integral case, certain three-dimensional domains are more easily described using an alternative set of coordinates. For example, a cylinder has rotational symmetry about one axis, which would be neatly captured in polar coordinates. Our goal is to extend our polar coordinates to the three-dimensional case.

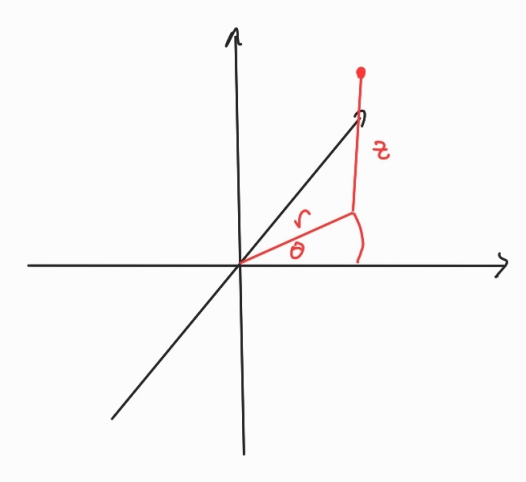


Figure 24: Cylindrical coordinates.

The easiest way to do this is as follows: starting with rectangular coordinates x, y, and z, replace our coordinates on the xy-plane with r and θ, and leave the z coordinate alone. This gives us **cylindrical coordinates**, which comes from the idea that cylinders are easily described using these coordinates. This comes with the change-of-coordinates formulas

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = z.$$

However, just as in the polar case, we need to be careful with dV. In fact, we have that

$$dV = r\,dr\,d\theta\,dz$$

which can be seen intuitively in the figure below.

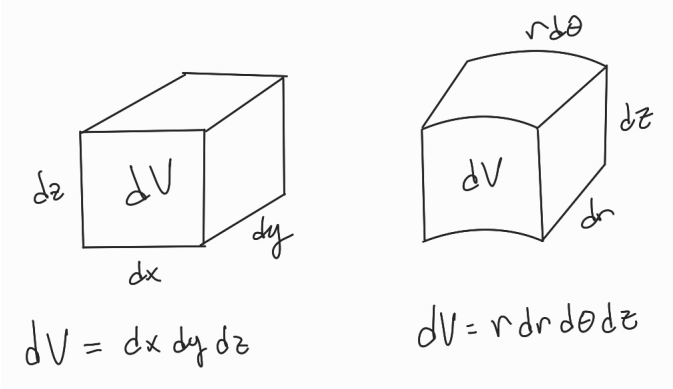


Figure 25: The volume of dV in cylindrical coordinates.

Since the volume should have dimensions of length^3, this does indeed make sense, as dr and dz each contribute one dimension of length, and $d\theta$ does not. We can also "derive" this algebraically (if you feel comfortable with the derivation for the polar case) by computing

$$dV = dx\,dy\,dz = (dA)\,dz = r\,dr\,d\theta\,dz$$

to obtain the same formula. This allows us to integrate over "cylindrically" symmetric regions in three dimensions more easily.

Example. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. The density at any point is proportional to its distance from the z-axis. Find the mass of E.

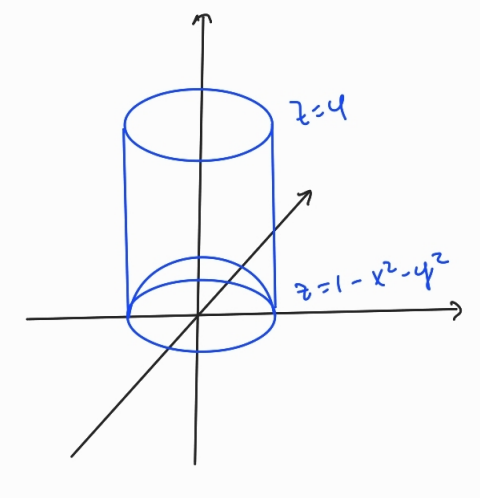


Figure 26: The region E.

We first address the density: since the distance to the z-axis is given by $\sqrt{x^2 + y^2} = r$, and the density is proportional to this distance, we have that $\sigma(x, y) = K\sqrt{x^2 + y^2}$, or $\sigma(r, \theta) = Kr$, for some constant K.

We notice that in polar coordinates, we parameterize over a circle, as r goes from 0 to 1 and θ goes from 0 to 2π. In the z direction, at constant r and θ, we see that z goes from $1 - x^2 - y^2$ to 4. However, we need to rewrite the lower bound in terms of r and θ, so we have that $1 - x^2 - y^2 = 1 - r^2$.

We then compute

$$\begin{aligned}
m &= \iiint_E dm \\
&= \int_0^1 \int_0^{2\pi} \int_{1-r^2}^4 \sigma(r, \theta) r\,dz\,d\theta\,dr \\
&= \int_0^1 \int_0^{2\pi} \int_{1-r^2}^4 Kr^2\,dz\,d\theta\,dr \\
&= \int_0^1 \int_0^{2\pi} Kr^2(3 + r^2)\,d\theta\,dr \\
&= 2\pi K \int_0^1 3r^2 + r^4)\,dr \\
&= 2\pi K \left(\frac{6}{5}\right) \\
&= \boxed{\frac{12\pi K}{5}}
\end{aligned}$$

to obtain the mass of E.

The reason that this problem works out so nicely in cylindrical coordinates is because it fits nicely into the coordinate system: the region of integration is naturally a cylinder, and so parameterizing it using cylindrical coordinates makes the bounds of integration easy. As you do more problems, you'll get a better feel for which coordinates work best for what sorts of regions.

Example. Let E be the region bounded by the plane $z = 2$ and the paraboloid $z = x^2 + y^2$, with density $\rho(x, y, z) = z^2$. Find the center of mass of z.

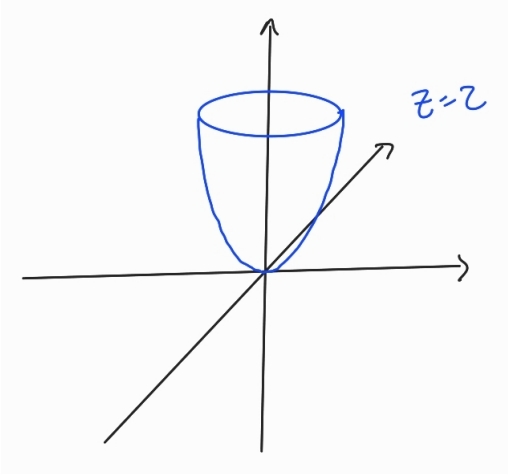


Figure 27: The region E.

We first parameterize the region. Since the "shadow" of the region is over the circle of radius 2 centered at the origin, we have that r goes from 0 to $\sqrt{2}$, and θ goes from 0 to 2π. At a constant value of r and θ, the variable z goes from $x^2 + y^2 = r^2$ to 2. We then have that

$$\begin{aligned} m &= \iiint_E dm \\ &= \int_0^{\sqrt{2}} \int_0^{2\pi} \int_{r^2}^{2} z^2 r \, dz \, d\theta \, dr \\ &= 2\pi \int_0^{\sqrt{2}} \frac{r}{3} \left(8 - r^6\right) \, dr \\ &= \frac{2\pi}{3} \int_0^{\sqrt{2}} 8r - r^7 \, dr \\ &= \frac{2\pi}{3} \left(4 \cdot 2 - \frac{16}{8}\right) \\ &= 4\pi \end{aligned}$$

which gives the mass of the region. Note that both the region and density is symmetric in both x and y, and thus both of these coordinates of the center of mass will be 0. Mathematically, both the integrals

$$\iiint_E x \, dm = \iiint_E xz^2 \, dV \qquad \iiint_E y \, dm = \iiint_E yz^2 \, dV$$

have integrands which are either odd in x or odd in y, and therefore will give 0, since the region is symmetric in both x and y. We compute the z-coordinate of the center of mass. We have

$$\iiint_E z \, dm = \iiint_E z^3 \, dm$$

$$
\begin{aligned}
&= \int_0^{\sqrt{2}} \int_0^{2\pi} \int_{r^2}^{2} z^3 r \, dz \, d\theta \, dr \\
&= 2\pi \int_0^{\sqrt{2}} \frac{r}{4} \left(16 - r^8\right) \, dr \\
&= \frac{\pi}{2} \int_0^{\sqrt{2}} 16r - r^9 \, dr \\
&= \frac{\pi}{2} \left(8r^2 - \frac{1}{10} r^{10} \Big|_0^{\sqrt{2}} \right) \\
&= \frac{\pi}{2} \left(16 - \frac{32}{10} \right) \\
&= \frac{32}{5} \pi
\end{aligned}
$$

and therefore the z-coordinate of the center of mass is given by

$$
\frac{\frac{32}{5}\pi}{4\pi} = \frac{8}{5}
$$

and we conclude that the center of mass of E is located at $\boxed{\left(0, 0, \frac{8}{5}\right)}$.

8 Triple Integrals in Spherical Coordinates

Textbook Reference: Chapter 15.8.
Planned Lecture Date(s): April 12, 2023.

In this section, we introduce yet another system of coordinates that might be helpful for us to parameterize regions. You can think of this as a further generalization of polar coordinates, but it works particularly well with regions that are spherically symmetric.

We make the following changes of coordinates. Let ρ denote the distance from our point (x, y, z) to the origin. Then, let φ be the angle between the vector pointing directly upwards (in the z-direction) to the vector (x, y, z), and let θ be the angle between the vector pointing in the x-direction and the vector (x, y) in the xy-plane.

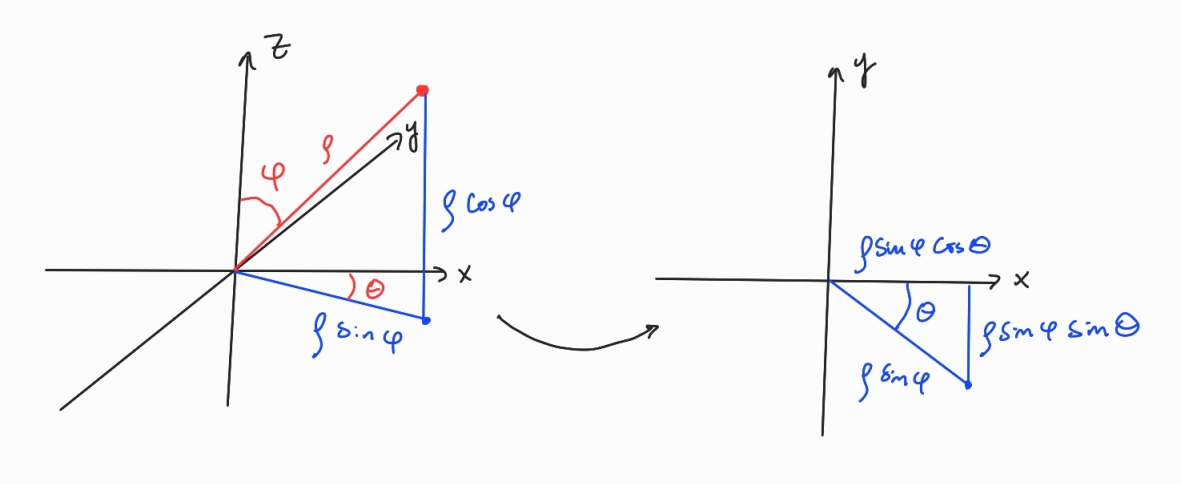


Figure 28: Spherical coordinates.

This corresponds to the coordinates:

$$x = \rho \sin(\varphi) \cos(\theta) \qquad y = \rho \sin(\varphi) \sin(\theta) \qquad z = \rho \cos(\varphi)$$

In particular, we notice that in order to range over a sphere of radius R, we keep $\rho = R$ fixed, and let θ range from 0 to 2π, but φ only ranges from 0 to π, as once it reaches the bottom, going further would correspond to a different choice of θ. These coordinates are also consistent with latitude and longitude, although latitude usually sets 0 to be at the equator rather than at the north pole.

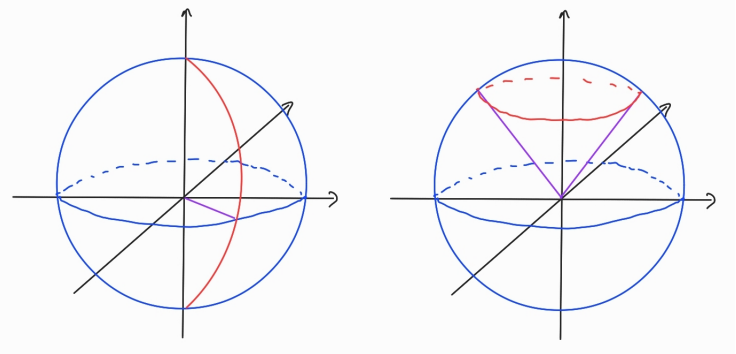


Figure 29: A line where φ varies from 0 to π, while ρ and θ are fixed (left), and a line where θ varies from 0 to 2π, while ρ and φ are fixed (right).

Note: In physics, the angles θ and φ are sometimes switched: φ will represent the angle in the xy-plane, and θ will represent the angle from the north pole. There is a good reason for this, as the common ordering (ρ, θ, φ) is then a right-handed coordinate system, as (locally) $\vec{\rho} \times \vec{\theta} = \vec{\varphi}$. However, the label of θ then disagrees with the definition from polar/cylindrical coordinates (whereas it does in the labelling that we have chosen).

Example. Parameterize the following regions in spherical coordinates. For the region on the left, the image refers to the region between the two spheres.

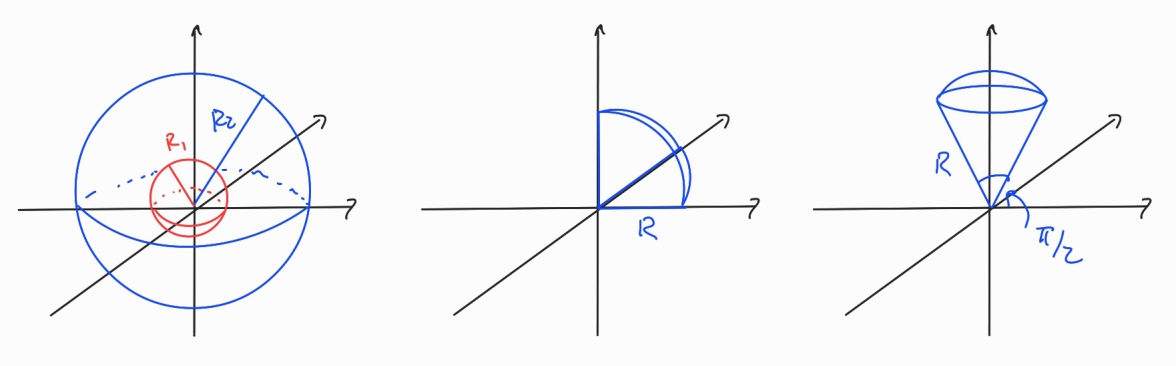


Figure 30: Regions to parameterize.

For the region on the left, we would have

$$R_1 \leq \rho \leq R_2 \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq \varphi \leq \pi$$

For the region in the middle, we would have

$$0 \leq \rho \leq R \qquad 0 \leq \theta \leq \frac{\pi}{4} \qquad 0 \leq \varphi \leq \frac{\pi}{2}$$

For the region on the right, we would have

$$0 \leq \rho \leq R \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq \varphi \leq \frac{\pi}{4}$$

Note that φ ranges to $\frac{\pi}{4}$, rather than $\frac{\pi}{2}$, since the angle should be measured from the z-axis down to the edge of the cone, rather than all the way across.

The next question to answer is how this choice of coordinates changes dV.

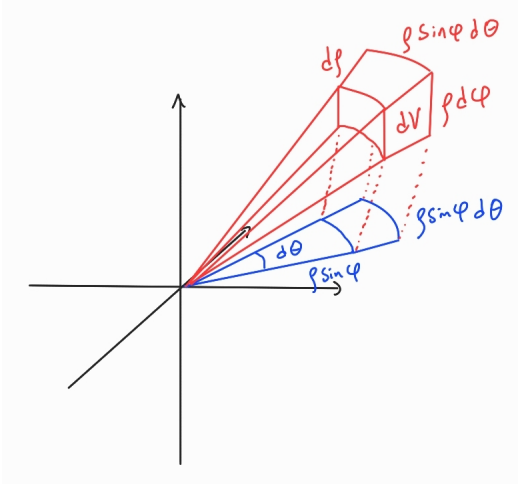


Figure 31: Finding the expression for dV in terms of $d\rho$, $d\varphi$, and $d\theta$.

The sides of our small box with volume dV are then given by $d\rho$, $\rho\, d\varphi$, and $\rho\sin(\varphi)\, d\theta$. We thus have that

$$dV = (d\rho)(\rho\, d\varphi)(\rho\sin(\varphi)\, d\theta) = \rho^2\sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

and this indeed matches our sanity check that this should have dimensions of length^3, as ρ^2 and $d\rho$ each contribute one factor of length, and the angles are dimensionless.

Example. Compute the volume of a sphere of radius R.

Let S denote the sphere. As before, we must integrate the constant function 1 over the domain of integration. Since our spherical region is parameterized by ρ varying from 0 to R, θ varying from 0 to 2π, and φ varying from 0 to π, we have that

$$\begin{aligned}
\text{Vol}(S) &= \iiint_S 1\, dV \\
&= \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2\sin(\varphi)\, d\varphi\, d\theta\, d\rho \\
&= 2\pi \left(\int_0^R \rho^2\, d\rho\right)\left(\int_0^{\pi} \sin(\varphi)\, d\varphi\right) \\
&= 2\pi\left(\frac{1}{3}R^3\right)(2) \\
&= \boxed{\frac{4}{3}\pi R^3}
\end{aligned}$$

which agrees with the formulas we have been given in previous classes.

Example. Compute the moment of inertia of a sphere of radius R about the z-axis of constant density K.

We first notice that for any point (ρ, θ, φ), from the picture above we see that $r_\perp = \rho\sin(\varphi)$. We thus have that

$$\begin{aligned}
I_z &= \iiint_E r_\perp^2\, dm \\
&= \int_0^R \int_0^{2\pi} \int_0^{\pi} (\rho^2\sin^2(\varphi))K\rho^2\sin(\varphi)\, d\varphi\, d\theta\, d\rho \\
&= 2\pi K\left(\int_0^R \rho^4\, d\rho\right)\left(\int_0^{\pi} \sin^3(\varphi)\, d\varphi\right) \\
&= \frac{2}{5}R^5\pi K\int_0^{\pi} (1-\cos^2(\varphi))\sin(\varphi)\, d\varphi \\
&= \frac{2}{5}R^5\pi K\int_{-1}^{1} 1-u^2 \qquad (u = \cos(\varphi)) \\
&= \frac{2}{5}R^5\pi K\left(u - \frac{1}{3}u^3\Big|_{-1}^{1}\right) \\
&= \frac{2}{5}R^5\pi K\left(\frac{4}{3}\right)
\end{aligned}$$

However, notice from the previous example that $V = \frac{4}{3}\pi R^3$, and since density is constant, we have that $m = KV = \frac{4}{3}\pi R^3 K$, and so we have

$$= \boxed{\frac{2}{5}mR^2}$$

which you may recognize from physics as the moment of inertia of a sphere.

9 Change of Variables in Multiple Integrals

Textbook Reference: Chapter 15.9.
Planned Lecture Date(s): April 14, 2023.

Recall from single-variable calculus that if we are computing an integral

$$\int_a^b f(x)\,dx$$

and we have a function $x = g(u)$, we can perform a u-substitution by computing $dx = g'(u)\,du$, and write

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $g(c) = a$ and $g(d) = b$. This is an example of a change of variables in the single-integral case, as we have changed from integrating over x to integrating over u. However, changing from x to u does not simply mean changing all instances of x in the integrand to the appropriate function of u, since we also need to change dx to the appropriate function of du. In the case of one variable, we simply attach $g'(u)$ to du, but in several variables, we need to be more careful.

In the case of two variables, let's suppose that we change our variables from x and y to u and v via the functions

$$x = f(u, v) \qquad y = g(u, v).$$

In order for this to be a change of variables, both f and g need to be one-to-one (different inputs go to different values) and continuous, to ensure that the change of variables is invertible. Then, recall via the chain rule that we have

$$dx = \frac{\partial f}{\partial u}\,du + \frac{\partial f}{\partial v}\,dv \qquad dy = \frac{\partial g}{\partial u}\,du + \frac{\partial g}{\partial v}\,dv$$

and we can picture this intuitively in the diagram below.

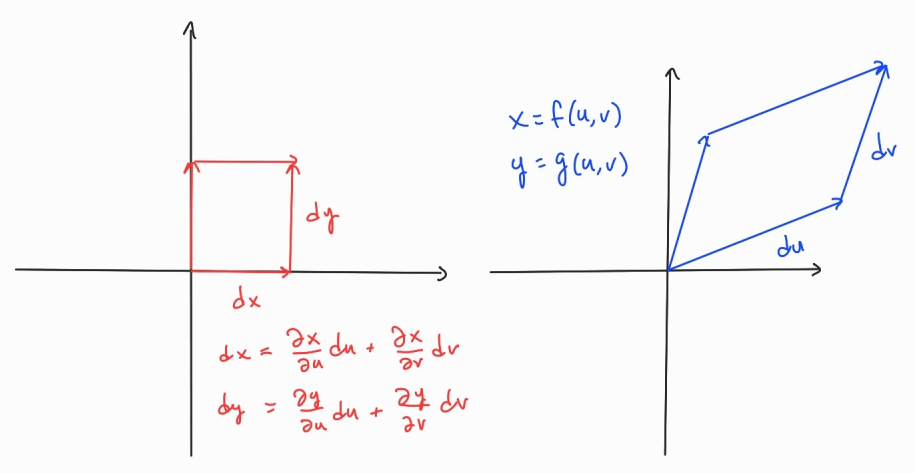


Figure 32: Changing from x and y to u and v.

If we think of dx, dy, du, and dv as vectors, then integrating over $dx\,dy$ is not the same thing as integrating over $du\,dv$, since these vectors could be of different magnitudes and directions and therefore determine different "chunks" of area. However, if we compare the amount that the area of $dx\,dy$ and $du\,dv$ differ by, we can simply account for this in our computation of an integral. For example, if the parallelogram that $dx\,dy$ defined in the above diagram had $\frac{1}{3}$ the area of $du\,dv$, we could integrate with respect to $du\,dv$, which

would cover three times as much area, and then multiply our result by $\frac{1}{3}$. This is the process we will try to emulate in general.

We can actually compute the area of $du\,dv$ using the determinant of a matrix. Given a set of n vectors in $\mathbb{R}^n$, the determinant of the matrix whose columns (or rows) are the n chosen vectors gives the n-dimensional volume of the parallelopiped (i.e., an n-dimensional box) that the vectors define. When $n = 2$, we can apply this to the vectors dx and dy, and from the chain rule above, we can write them as a matrix in du and dv as

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

and the determinant of this matrix will give the area of the parallelogram defined by dx and dy relative to the area of the parallelogram defined by du and dv. Thus, if we would like to integrate with respect to $du\,dv$ instead, we should multiply by this determinant, called the **Jacobian** of this change of variables. We denote the Jacobian by

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

and thus to compute our integral, we have that

$$dA = dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

and therefore

$$\iint_D h(x,y) = \iint_{D'} h(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where D' denotes the region that D is sent to under the change of variables.

We can do the same thing in 3 variables, by changing our variables via

$$x = x(u,v,w) \qquad y = y(u,v,w) \qquad z = z(u,v,w)$$

and computing the Jacobian by finding

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

where this determinant of a 3×3 matrix can be computed using various methods, such as expansion by minors or the "basket-weaving" method. We then compute

$$\iiint_E f(x,y,z)\,dV = \iiint_{E'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\,dw$$

to change variables in three dimensions.

Note: You may notice this pesky absolute value surrounding our Jacobian term - this corresponds to the idea that when we change variables, the determinant will keep track of the right or left handedness of our new coordinate system. If we change the chirality (i.e., right to left or left to right) of our coordinate system when changing variables, the Jacobian determinant will be negative. All of your integrals will go through exactly the same way, except possibly with the wrong choice of sign (think about how integrals keep track of signed area). One way that this can be resolved is to change the order of the new coordinate variables (i.e. using (v,u,w) rather than (u,v,w)). This delves into the concept of orientation, which is beyond the scope of this class, but for all practical purposes in this course, the sign of the Jacobian will not matter.

Additional Note: For those curious, you may notice that the Jacobian terms in polar, cylindrical, and spherical coordinates are r, r, and $\rho^2 \sin(\varphi)$ are actually always non-negative. Since $r, \rho > 0$ and $\varphi \in [0, \pi]$, these Jacobian terms are always at least 0, which is good, because it means that the coordinate system does not suddenly change chirality at specific values. However, something interesting does occur when the Jacobian is 0, which happens when either $r = 0$, $\rho = 0$, or if $\varphi = 0$ or $\varphi = \pi$. What points do these correspond to, and what happens to the coordinate system at these points?

Now is a good time to formalize all of the previous changes of coordinates we have been using, and to verify that the Jacobian in each of those integrals was indeed the correct choice. For polar coordinates, we have that

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

and so we have

$$\begin{aligned}\frac{\partial(x,y)}{\partial(r,\theta)} &= \det\begin{pmatrix}\cos(\theta) & -r\sin(\theta)\\ \sin(\theta) & r\cos(\theta)\end{pmatrix}\\ &= r\cos^2(\theta) + r\sin^2(\theta)\\ &= r\end{aligned}$$

which does indeed agree with the Jacobian that we have been using. We do the same for cylindrical coordinates.

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = z$$

We then have

$$\begin{aligned}\frac{\partial(x,y,z)}{\partial(r,\theta,z)} &= \det\begin{pmatrix}\cos(\theta) & -r\sin(\theta) & 0\\ \sin(\theta) & r\cos(\theta) & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= r\cos^2(\theta) + r\sin^2(\theta)\\ &= r\end{aligned}$$

and this again agrees with our previous Jacobian. Finally, we check spherical coordinates.

$$x = \rho\sin(\varphi)\cos(\theta) \qquad y = \rho\sin(\varphi)\sin(\theta) \qquad z = \rho\cos(\varphi)$$

We have

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \det\begin{pmatrix}\sin(\varphi)\cos(\theta) & -\rho\sin(\varphi)\sin(\theta) & \rho\cos(\varphi)\cos(\theta)\\ \sin(\varphi)\sin(\theta) & \rho\sin(\varphi)\cos(\theta) & \rho\cos(\varphi)\sin(\theta)\\ \cos(\varphi) & 0 & -\rho\sin(\varphi)\end{pmatrix}$$

Expanding by minors along the bottom row (since there is a 0 to save us some work), we compute

$$\begin{aligned}&= \cos(\varphi)\Big(-\rho^2\sin(\varphi)\cos(\varphi)\sin^2(\theta) - \rho^2\sin(\varphi)\cos(\varphi)\cos^2(\theta)\Big)\\ &\quad - \rho\sin(\varphi)\Big(\rho\sin^2(\varphi)\cos^2(\theta) + \rho\sin^2(\varphi)\sin^2(\theta)\Big)\\ &= \cos(\varphi)\Big(-\rho^2\sin(\varphi)\cos(\varphi)\Big) - \rho\sin(\varphi)\Big(\rho\sin^2(\varphi)\Big)\\ &= -\rho^2\sin(\varphi)\cos^2(\varphi) - \rho^2\sin(\varphi)\sin^2(\varphi)\\ &= -\rho^2\sin(\varphi)\end{aligned}$$

which *almost* agrees with our Jacobian from before, but remembering to take an absolute value shows that we indeed are using the right Jacobian term. This minus sign tells us that our choice of (ρ, θ, φ) is in fact a left-handed coordinate system, which gives the physicists labelling of spherical coordinates (as discussed before) a bit more merit.

Example. Let R be the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$. Evaluate the integral

$$\iint_R e^{\frac{x+y}{x-y}}\,dA$$

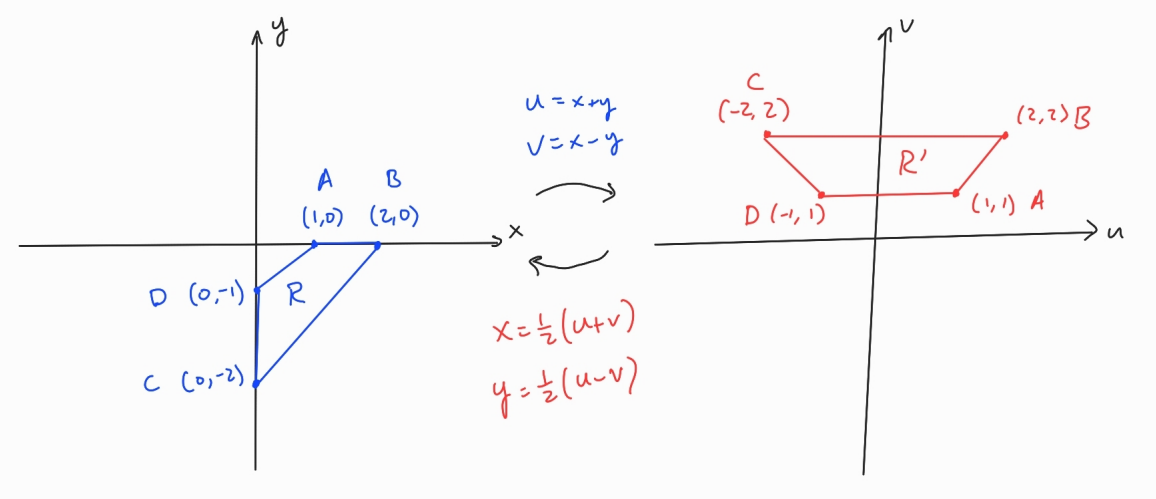


Figure 33: The regions in the xy-plane (left) and the uv-plane (right).

We make the change of coordinates

$$u = x + y \qquad v = x - y$$

which we can rewrite in x and y as

$$x = \frac{1}{2}(u+v) \qquad y = \frac{1}{2}(u-v)$$

We then compute the Jacobian to be

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}$$

From the picture, we see that in R', we have that v goes from 1 to 2, and at a constant v, u goes from $-v$ to v. Thus, we have that

$$\begin{aligned}
\iint_R e^{\frac{x+y}{x-y}}\,dx\,dy &= \iint_{R'} e^{\frac{u}{v}} \left|\frac{1}{2}\right| du\,dv \\
&= \frac{1}{2}\int_1^2 \int_{-v}^{v} e^{\frac{u}{v}}\,du\,dv \\
&= \frac{1}{2}\int_1^2 \left(v e^{\frac{u}{v}}\Big|_{-v}^{v} \right) dv \\
&= \frac{1}{2}\int_1^2 v(e - e^{-1})\,dv \\
&= \frac{1}{2}(e-e^{-1})\int_1^2 v\,dv \\
&= \frac{1}{2}(e-e^{-1})\frac{3}{2} \\
&= \boxed{\frac{3}{4}(e-e^{-1})}.
\end{aligned}$$

10 Chain Rule

Textbook Reference: Chapter 14.5.
Planned Lecture Date(s): April 17, 2023.

From single variable calculus, we have that if $y = f(x)$, and $x = x(t)$ is a function of some variable t, we can relate their derivatives using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Formally, we would need to be a bit more careful writing this, since we'd have to plug $x = x(t)$ into $\frac{dy}{dx}$, but in the way we have written it, we can illustrate that the chain rule does feel intuitive: if we treat the derivatives as fractions, then they would cancel to give the result we want.

Now let's suppose that $f(x, y)$ is a function of two variables x and y, and additionally $x = x(t)$ and $y = y(t)$ are both functions of another variable t. How does f change when we change t?

Let's first answer the question of how f changes when we change x and y. Independently, we have that when x changes by Δx, f changes by $\frac{\partial f}{\partial x}\Delta x$, and when y changes by Δy, f changes by $\frac{\partial f}{\partial y}\Delta y$.

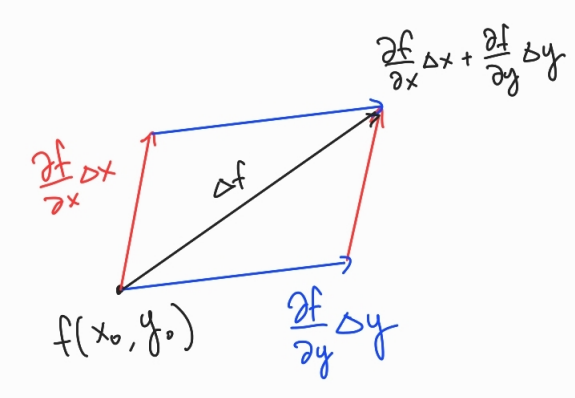


Figure 34: The change in f coming from a change in x and a change in y.

Thus, f should change by the sum of the changes in the x and y direction. Thus, we write that

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

and so the derivatives should follow a similar rule.

Then, since x and y both change with respect to t by $\frac{\partial x}{\partial t}$ and $\frac{\partial y}{\partial t}$ respectively, we write

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

to obtain (one formulation of) the **chain rule**. Note that if we want to actually evaluate this, we need to substitute x and y in our resulting expression with the corresponding functions of t.

If x and y are instead functions of two variables s and t, we can compute the partial derivatives using the same rule:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

We'd like to be able to compute partial derivatives of multivariable functions in general - it turns out that knowing all of the partial derivatives allows us to know all of the "differential" information of this function, and this idea will be formalized in the next lecture using the gradient.

This motivates us to construct a general formula: if $f(x_1, \ldots, x_n)$ is a function of n variables x_i, and $x_i = x_i(t_1, \ldots, t_m)$ is a function of m variables t_j, then we have that

$$\frac{\partial f}{\partial t_j} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_j}$$

which is the general statement of the chain rule.

Example. Let $z = x^2y + 3xy^4$, and let $x = \sin(2t)$ and $y = \cos(t)$. Find $\frac{dz}{dt}$ when $t = 0$.

We first note that x at $t = 0$ is 0, and y at $t = 0$ is 1. We compute

$$\begin{aligned}\left.\frac{dz}{dt}\right|_{t=0} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= (2xy + 3y^4)(2\cos(2t)) + (x^2 + 12xy^3)(-\sin(t)) \\ &= (0 + 3)(2) + (0 + 12)(0) \\ &= \boxed{6}\end{aligned}$$

Example. Let $z = e^x \sin(y)$, and let $x = st^2$ and $y = s^2t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

We compute

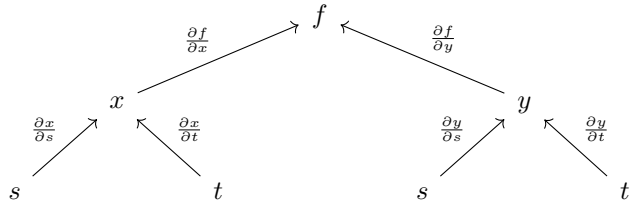
$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= (e^x \sin(y))(t^2) + (e^x \cos(y))(2st) \\ &= t^2 e^{st^2}\sin(s^2t) + 2ste^{st^2}\cos(s^2t) \\ &= \boxed{te^{st^2}\left(t\sin(s^2t) + 2s\cos(s^2t)\right)}\end{aligned}$$

and similarly

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= (e^x \sin(y))(2st) + (e^x \cos(y))(s^2) \\ &= 2ste^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t) \\ &= \boxed{se^{st^2}\left(2t\sin(s^2t) + s\cos(s^2t)\right)}\end{aligned}$$

which gives the desired partial derivatives.

One way that you can remember which partial derivatives to sum is using the tree method, which can be summarized as follows. First, draw the tree which corresponds to the relationships between the variables. Then, in order to find how f changes with respect to a variable on the bottom row, for example s, find all paths from f to s, and multiply the partial derivatives along the path. Finally, add up all the products you have obtained from each distinct way to get from f to s.



This method generalizes to any number of variables (or intermediate "layers"), so you can compute very complicated partial derivatives using the chain rule and this technique.

We can also use the chain rule to perform implicit differentiation. Recall from single variable calculus that if we have a function $f(x, y) = 0$, we have a method for computing $\frac{dy}{dx}$ which involves manipulating the chain rule in a clever way. We can perform a similar trick here: suppose $F(x, y, z) = 0$ implicitly defines z as a function of x and y. Suppose we would like to solve for $\frac{\partial z}{\partial x}$, where y is held constant. Then, we have that

$$\begin{aligned} F(x, y, z) &= 0 \\ \frac{\partial}{\partial x} F(x, y, z) &= \frac{\partial}{\partial x} 0 \\ \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} &= 0 \end{aligned}$$

Here, we notice that $\frac{\partial y}{\partial x} = 0$, since y is held constant, and $\frac{\partial x}{\partial x} = 1$, so we have

$$\begin{aligned} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial x} &= -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \end{aligned}$$

which gives us $\frac{\partial z}{\partial x}$ solely as a function of derivatives of F. This holds more generally with n variables (but is derived in exactly the same way), as if we have $F(x_1, \ldots, x_n)$, then

$$\frac{\partial x_i}{\partial x_j} = -\frac{F_j}{F_i}$$

where the partial derivative on the left is taken with all x_k held constant, when $k \neq i, j$. Note that F_{x_k} is shorthand for $\frac{\partial F}{\partial x_k}$.

Example. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

We define $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then, the function which implicitly defines z as a function of x or y is given by $F(x, y, z) = 0$, so we apply our formula. We have that

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = \boxed{-\frac{x^2 + 2yz}{z^2 + 2xy}}$$

and similarly

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = \boxed{-\frac{y^2 + 2xz}{z^2 + 2xy}}$$

which gives our desired answers.

11 Directional Derivatives and the Gradient Vector

Textbook Reference: Chapter 14.6.
Planned Lecture Date(s): April 24, 2023.

Recall from Math 126 that given a function $f(x, y)$, we can compute a directional derivative, which corresponds to the slope of the line which best approximates the function in a specific direction.

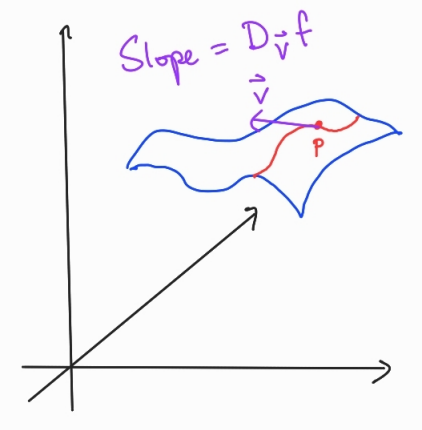


Figure 35: Intuition for the directional derivative.

More precisely, given a function $f(x, y)$, a point (x_0, y_0), and some vector $\vec{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$ with unit length (i.e. $\|\vec{v}\| = 1$), the directional derivative $D_{\vec{v}} f$ is given by the formula

$$(D_{\vec{v}} f)(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0)$$

and this is the slope of the line which is tangent to f at (x_0, y_0), going in the direction of $\vec{v}$.

You may notice that this looks suspiciously like a dot product, and in fact, it is! We can write

$$D_{\vec{v}} f = \begin{pmatrix} f_x(x, y) & f_y(x, y) \end{pmatrix} \cdot \vec{v}$$

The vector on the left turns out to be incredibly useful, and so we define the **gradient** of f, often denoted ∇f, to be the vector

$$\nabla f = \begin{pmatrix} f_x(x, y) & f_y(x, y) \end{pmatrix}$$

The symbol used to denote the gradient is frequently called "del" or "nabla". It turns out that this vector is incredibly useful to us, and will show up in lots of different places! For a function $g(x, y, z)$ of three variables, we similarly define

$$\nabla g = \begin{pmatrix} g_x(x, y, z) & g_y(x, y, z) & g_z(x, y, z) \end{pmatrix}$$

as the gradient in three dimensions. You should think of the gradient as a tool which takes in a function (which outputs numbers at every point) and returns a vector (or more precisely, a function which returns vectors at every point).

One such question we can now ask is: given some function $f(x, y)$ and a point (x_0, y_0), in what direction should I move to maximize how much $f(x, y)$ changes? We can rephrase this question more mathematically as the following: given f and (x_0, y_0), what (unit) vector $\vec{v}$ will increase f the most? This can be interpreted naturally as asking for which choice of $\vec{v}$ maximizes $D_{\vec{v}} f$. We have that

$$D_{\vec{v}} f = \nabla f \cdot \vec{v} = |\nabla f||\vec{v}| \cos(\theta) = |\nabla f| \cos(\theta)$$

where θ denotes the angle between ∇f and $\vec{v}$. This quantity is maximized when $\cos(\theta) = 1$, or when $\theta = 0$, in which case $\vec{v}$ is in the same direction as ∇f.

This leads to some interesting observations: the first is that ∇f represents the direction at each point in which to move to increase f the most. Additionally, if (x_0, y_0) is a critical point, $(\nabla f)(x_0, y_0) = 0$. This is similar to the role of the derivative in single-variable calculus, as $\frac{df}{dx} = 0$ occurs precisely when f has a critical point. In fact, many properties of ∇f are shared with the derivative from single-variable calculus, and you should think of ∇f as an analogue of the derivative in higher dimensions.

Example. Let $f(x, y) = x^2y^3 - 4y$ and let $p = (2, -1)$. Find the direction $\vec{v}$ at p which maximizes the change in f.

We compute

$$\nabla f = 2xy^3\hat{\mathbf{i}} + (3x^2y^2 - 4)\hat{\mathbf{j}}$$

and evaluating at $(2, -1)$, we obtain the vector

$$\nabla f\Big|_{(2,-1)} = \boxed{-4\hat{\mathbf{i}} + 8\hat{\mathbf{j}}}$$

and this gives the direction of maximum change (you can normalize if you want).

With this in mind, let's think about tangent planes to surfaces. Let $f(x, y, z) = c$ be some surface in three dimensions, and pick some point (x_0, y_0, z_0). Then, the tangent plane of f at (x_0, y_0, z_0) should be the plane which best approximates f at this point, so we should try to find vectors which don't try to "leave" f near (x_0, y_0, z_0). Mathematically, this equates to finding vectors $\vec{v}$ such that $D_{\vec{v}}f = 0$, since if we try to travel in the $\vec{v}$ direction, we'll find that we do not change f (i.e., f changes by $D_{\vec{v}}f = 0$).

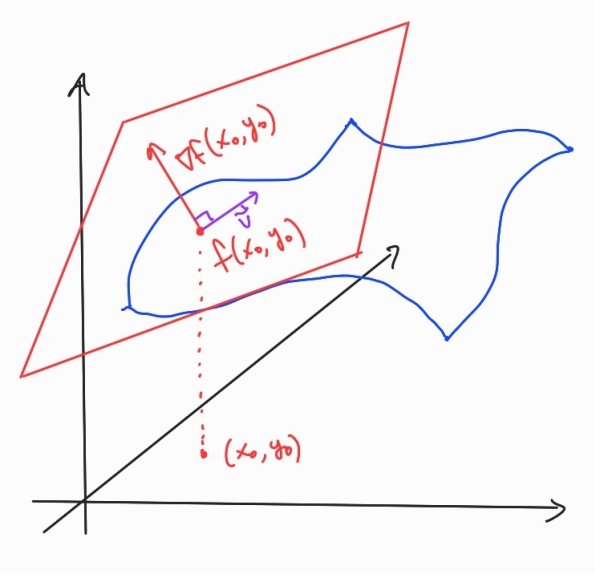


Figure 36: Vectors lying in the tangent plane.

We are therefore looking for vectors $\vec{v}$ such that

$$0 = D_{\vec{v}}f = |\nabla f| \cdot \vec{v}$$

and we recall that vectors have dot product equal to 0 precisely when they are orthogonal. We conclude first that any vector in the tangent plane must satisfy the above equation, and so for any point (x, y, z), the

vector $\vec{v} = \begin{pmatrix} x - x_0 & y - y_0 & z - z_0 \end{pmatrix}$ is the vector from (x_0, y_0, z_0) to (x, y, z), and so $\vec{v}$ is in the tangent plane if and only if

$$0 = |\nabla f| \cdot \vec{v} = f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

which gives the equation of the plane. This also tells us that the gradient vector has a dot product of 0 to every vector in the tangent plane, and thus represents the normal vector to the plane at (x_0, y_0, z_0).

Example. Find the normal vector and the equation of the tangent plane to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$ at the point $(-2, 1, -3)$.

We define $F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$. Then, we compute

$$F_x = \frac{1}{2}x \qquad F_y = 2y \qquad F_z = \frac{2}{9}z$$

and evaluating at $(-2, 1, -3)$ we have that

$$F_x = -1 \qquad F_y = 2 \qquad F_z = -\frac{2}{3}$$

Thus, the normal vector is given by

$$\boxed{\nabla f = \begin{pmatrix} -1 & 2 & -\frac{2}{3} \end{pmatrix}}$$

and the equation of the tangent plane is given by

$$\boxed{-(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0}$$

Note: If you've taken a linear algebra course, you may commonly see vectors written as a column, whereas I have written the gradient as a row vector throughout this section. This decision is intentional, as the gradient is formally a "covector", but this discussion is beyond the scope of this course. Informally, this is because we (mostly) use the gradient to take dot products with other vectors (as evidenced in this chapter), and this makes the matrix multiplication work (as a 1×3 vector times a 3×1 vector gives a 1×1 vector, or scalar). However, for the purposes of this course, no understanding will be lost if you choose to think of it as a column vector.

12 Vector Fields

Textbook Reference: Chapter 16.1.
Planned Lecture Date(s): April 26, 2023.

Let's take a step back and think about what a function is. If I have a function $F(x, y, z)$ of real variables, then if I plug in three real numbers x, y, and z, $F(x, y, z)$ will output a real number, which I think of as the output of the function. If we instead interpret the ordered triple (x, y, z) as a point in three-dimensional space, then we can think of F as a function which takes in a point and outputs a number. In other words, F assigns a value to every point in space (and we can visualize this using its graph).

What if F assigned something else to each point? Let's instead suppose F assigned a vector to each point, rather than a number (or scalar). Then, at each point, F would assign a direction, and we can visualize this by drawing an arrow at each point. This is called a **vector field**.

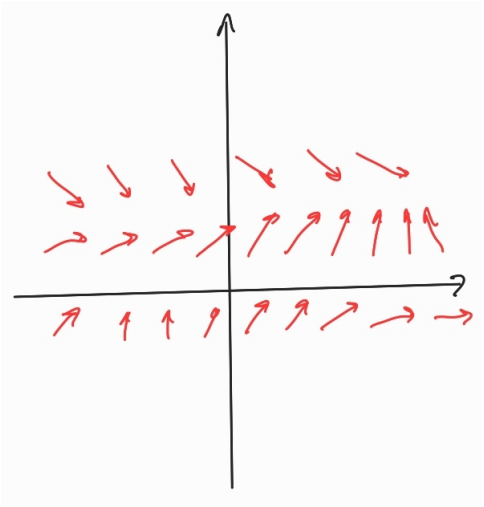


Figure 37: A sketch of a vector field in two dimensions.

Vector fields certainly have physical applications: for example, a vector field could represent the flow wind in space, the pull of gravity, or an electric or magnetic field. We can think of vector fields as several functions packaged into one vector, where each function tells us how much of each unit vector we should have. In other words, we can write

$$F(x, y, z) = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}$$

where P, Q, and R are each (scalar-valued) functions of x, y, and z. We can then define some notions of vector fields that we'd normally associate to scalar-valued functions, such as continuity: a vector field is continuous if P, Q, and R are all continuous.

Note: We will be using vectors extensively in this class, and often times variables can represent vectors or scalars. To emphasize the difference, please make an effort to put a "vector hat" on top of all of your variables which represent vectors, to differentiate them from scalars (i.e. $\vec{x}$ rather than x). This will save you tremendous headache during the rest of the course.

Example. Suppose the wind on the xy-plane is given by

$$F(x, y) = e^{-x^2-y^2}\hat{\mathbf{i}} + \sin(x + y)\hat{\mathbf{j}} + \arctan(yx^2)\hat{\mathbf{k}}$$

and I'd like to fly a flag in the $\hat{\mathbf{i}}$ direction somewhere in the plane. Where should I plant my flag to maximize the wind in the same direction as my flag?

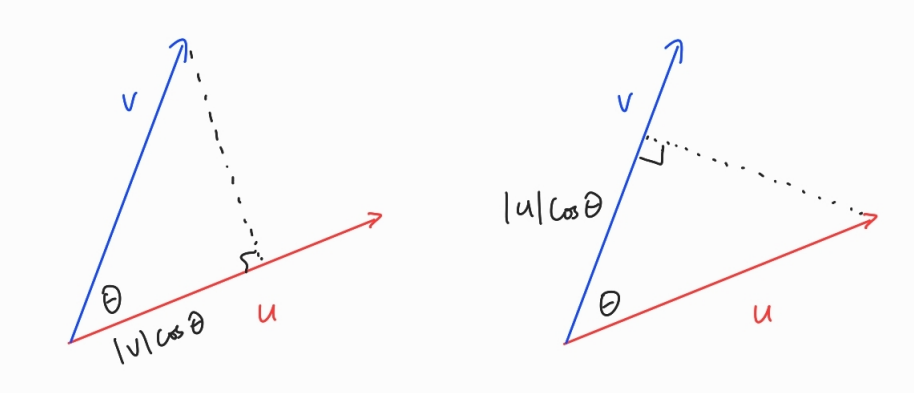


Figure 38: Symmetry of dot product, as $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta)$, where θ is the angle between $\vec{u}$ and $\vec{v}$.

Recall that the dot product of two vectors $\vec{u}$ and $\vec{v}$ measures the amount of $\vec{u}$ that "aligns" with $\vec{v}$ (or alternatively, the amount of $\vec{v}$ that "aligns" with $\vec{u}$). Thus, if I'd like to know how F aligns with $\hat{\mathbf{i}}$, I can simply compute the dot product $F(x, \vec{y}, z) \cdot \hat{\mathbf{i}}$. We compute this to be

$$F(x, \vec{y}, z) \cdot \hat{\mathbf{i}} = e^{-x^2-y^2}$$

and so this quantity tells us that at any point (x, y), the amount of $\vec{F}$ in the $\hat{\mathbf{i}}$ direction is given by $e^{-x^2-y^2}$. Note that since $\hat{\mathbf{i}}$ is a unit vector (i.e., $|\hat{\mathbf{i}}| = 1$), the dot product gives $|\vec{F}| \cos(\theta)$, which is precisely a measurement of how much of F is in the $\hat{\mathbf{i}}$ direction.

We can then see that this function is maximized at $\boxed{(0, 0)}$, where it has a value of 1.

Note: Understanding the use of the dot product will be incredibly important for the remainder of the course. Please make sure you understand the idea that the dot product measures how much a vector (or vector field) "aligns" with another.

One example of a vector field that we have already seen is the gradient: given some function $F(x, y, z)$, we can compute

$$\nabla F = \frac{\partial F}{\partial x}\hat{\mathbf{i}} + \frac{\partial F}{\partial y}\hat{\mathbf{j}} + \frac{\partial F}{\partial z}\hat{\mathbf{k}}$$

and since F is a function, all of these partial derivatives will be functions which depend on (x, y, z), and therefore define a vector at every point. Recall from our previous lecture that ∇F corresponds to the direction of greatest change in F - thus, the gradient of F gives a vector field which tells us the direction of greatest change in F, and we additionally have a formula for it, given above.

Note: We can intuitively think of the symbol ∇ to be the vector

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$$

This, of course, does not actually make sense, since the coefficients of the unit vectors are differential operators rather than scalars or functions. However, it is a useful bookkeeping tool for us, since if we have a scalar valued function F times a vector, we can write

$$\nabla F = \frac{\partial}{\partial x}F\hat{\mathbf{i}} + \frac{\partial}{\partial y}F\hat{\mathbf{j}} + \frac{\partial}{\partial z}F\hat{\mathbf{k}}$$

which is precisely how the gradient is defined. We'll see other examples later on where it's useful to think of ∇ as this vector.

A vector field which arises as the gradient of some function is called a **conservative** vector field. When this is the case, and we can write $F = \nabla f$ for some scalar-valued function f, the function f is referred to as the **potential function** for F. Note that if

$$F = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}$$

is indeed conservative, we have that

$$F(x, y, z) = \nabla f = \frac{\partial F}{\partial x}\hat{\mathbf{i}} + \frac{\partial F}{\partial y}\hat{\mathbf{j}} + \frac{\partial F}{\partial z}\hat{\mathbf{k}}$$

and by Clairaut's Theorem, which states that mixed partial derivatives commute, we see that

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

and so this (together with the symmetric choices of P, Q, and R) gives us a necessary condition for F to be conservative, as all of these equalities must hold. It turns out that this condition is sufficient with a few additional assumptions, which we will discuss later.

Example. Let the following be vector fields.

- $F(x, y) = (x^2 - y^2)\hat{\mathbf{i}} + (2xy)\hat{\mathbf{j}}$
- $G(x, y) = (x^2 + y^2)\hat{\mathbf{i}} + (2xy)\hat{\mathbf{j}}$

For each vector field, determine if it is conservative, and if it is, find a potential function.

We have the following.

- We check that
$$\frac{\partial}{\partial y}(x^2 - y^2) = -2y \neq 2y = \frac{\partial}{\partial x}(2xy)$$
and therefore $\boxed{F \text{ is not conservative}}$.

- We check that
$$\frac{\partial}{\partial y}(x^2 + y^2) = 2y = \frac{\partial}{\partial x}(2xy)$$
and so this could be conservative. If $P = \frac{\partial f}{\partial x}$, then we can recover f by computing $\int P\, dx$ (up to a constant in y). Since $Q = \frac{\partial f}{\partial y}$, we can then recover the constant in y by computing $\int Q\, dy$ (up to a constant). We integrate
$$\int x^2 + y^2\, dx = \frac{1}{3}x^3 + xy^2 + C_1(y)$$
and similarly
$$\int 2xy\, dy = xy^2 + C_2(x)$$
We see that these are indeed compatible, and so our potential function for G is given by
$$\boxed{g(x, y) = \frac{1}{3}x^3 + xy^2 + C}$$
for some real constant C (which does not depend on x and y), and so G is indeed conservative.

Note: The potential function derives its name from physics, where the electric (or gravitational) field is the (negative) gradient of the electric (or gravitational) potential. The minus sign comes from the fact that the gradient points in the direction of maximum change (and thus wants to increase), whereas physical objects want to move in the direction which decreases their potential most.

13 Line Integrals

Textbook Reference: Chapter 16.2.
Planned Lecture Date(s): April 28, 2023.

Thus far, all of our integrals have occurred over "flat" space: even though our bounds may be messy, we simply set bounds on our variables x, y, and z, and integrate over these bounds one by one. Today, we'll define a different integral, the **line integral**, that allows us to handle more interesting cases.

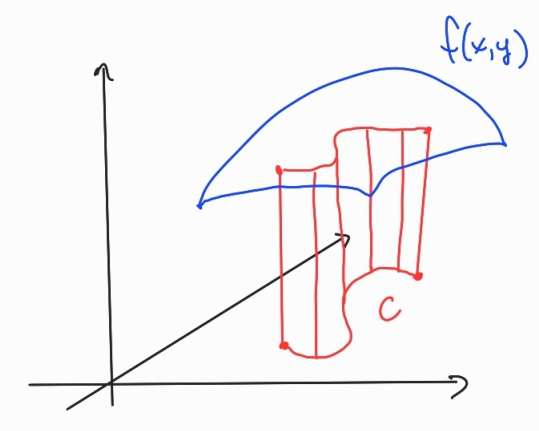


Figure 39: The line integral of f over the curve C.

Instead of integrating over some segment of the x-axis, I can instead choose some curve C in the xy-plane, and ask for the integral of some function over that curve. Geometrically, we can think of this as the area of the "curtain" over C, which follows the curve C, but "hangs" from the graph of $f(x, y)$. Even though we are only integrating over a one-dimensional object, the curve C, the fact that C really sits inside of two dimensional space plays an important role in the theory of integration.

Suppose we are able to parameterize C; in other words, we can find functions $x(t)$ and $y(t)$ such that when t varies inside the interval $[a, b]$, $x(t)$ and $y(t)$ will precisely follow the curve C. Let s denote the variable which represents how far along C we have travelled (this is not super precisely defined here, and mostly serves the purpose for the following reasoning). In order to integrate along C, we would like to integrate with respect to s. Then, a small change in s comes from a small change in x and a small change in y. However, these quantities are all scalars, and so they relate using the Pythagorean theorem.

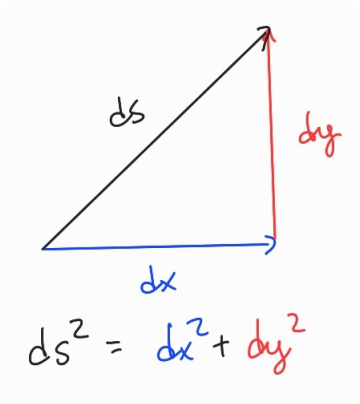


Figure 40: Relating dx, dy, and ds.

This motivates the claim that

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

and therefore if we would like to integrate over C, we can parameterize C via $x(t)$ and $y(t)$, and integrate as follows.

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The intuition here is that t travels from a to b, which forces x and y to travel along the curve C, and this integrates over the entire curve C. The square root term introduced when we change from ds to dt can be thought of as accounting for how the curve distorts length relative to t, in a similar way to how a Jacobian accounts for changes in dA or dV.

Example. Let C be the upper half of the unit circle. Compute

$$\int_C 2 + x^2 y\, ds$$

Note first that we can parameterize C using

$$x(t) = \cos(t) \qquad y(t) = \sin(t)$$

and let t vary within $[0, \pi]$. We therefore compute

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt = \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt = \, dt$$

and so we have

$$\begin{aligned}
\int_C 2 + x^2 y\, ds &= \int_0^\pi 2 + \cos^2(t)\sin(t)\, dt \\
&= 2\pi + \int_0^\pi \cos^2(t)\sin(t) \\
&= 2\pi + \int_{-1}^1 u^2\, du \qquad (u = -\cos(t)) \\
&= \boxed{2\pi + \frac{2}{3}}
\end{aligned}$$

In the special case where C is simply a horizontal or vertical line, we have that $dy = 0$ and $dx = 0$ respectively. This simplifies our expression for ds considerably, as we then have that

$$ds = y'(t)\, dt \qquad ds = x'(t)\, dt$$

respectively. Thus, if $dy = 0$, then $dx = ds$, and we write

$$\int_C f(x,y)\, dx = \int_C f(x(t), y(t)) x'(t)\, dt$$

and similarly when $dx = 0$, we have $dy = ds$, and so we write

$$\int_C f(x,y)\, dy = \int_C f(x(t), y(t)) y'(t)\, dt$$

which considerably simplifies our integrals, and will be incredibly useful to us later on, as we will frequently deal with expressions of the form

$$\int_C P\, dx + Q\, dy$$

which can be integrated separately using the above formulas.

We can extend all of these notions to curves lying in three dimensions, using a parameterization

$$x = x(t) \qquad y = y(t) \qquad z = z(t)$$

and using

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

to integrate over a parameterized curve. Just as we could over two and three-dimensional regions, we can find mass, centers of mass, and moments of inertia over one-dimensional curves living inside of three-dimensional space.

Note: Line integrals are also often called path integrals or contour integrals.

Line integrals are actually more powerful than what we've done so far. We can actually integrate quantities pertaining to vector fields, which opens up many new possibilities for things we can do with line integrals. The motivating example comes from the concept of work, from physics.

Intuitively, work is given by adding up the force applied over a distance, and frequently the first formulation of this we see is $W = F\Delta x$. However, this assumes that the applied force is constant, but in Calc 2, we may have previously seen the expression

$$W = \int F\, dx$$

where the (possibly variable) force is applied in a line, and the scalar quantity W measures the total force applied over a distance. Using line integrals, we can actually enhance this further: instead of applying the force in the same line as our direction of travel, we can travel in a different direction than the direction of our force, and keep track of the work done along the way.

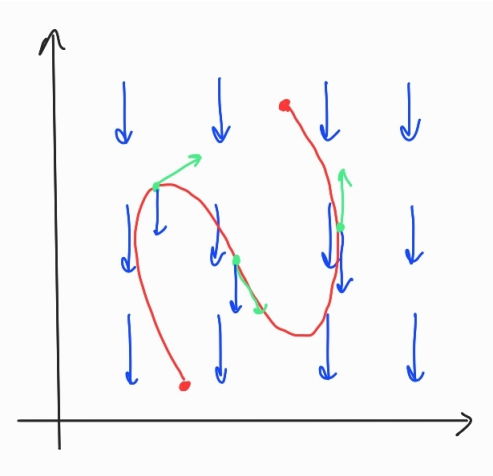


Figure 41: Finding the work done by the blue vector field over the red curve. At each of the green points, we take a dot product between $\vec{dr}$, the instantaneous motion of the curve (in green), with $\vec{F}$ (in blue), the force applied.

To do this, let $F(x, y, z)$ denote the force applied at any given point (x, y, z), given by

$$F(x, y, z) = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}$$

Then, let C be a smooth curve in three-dimensional space, and parameterize it using

$$\vec{r}(t) = \begin{pmatrix} x(t) & y(t) & z(t) \end{pmatrix}$$

where $\vec{r}(t)$ follows C. We should first observe that if the force $\vec{F}$ is applied not in the direction of motion, then the part of $\vec{F}$ in the same direction as the motion is the only part which contributes to the work. In other words, if $\vec{v}$ is a unit vector in the direction of motion, then $\vec{F} \cdot \vec{v}$ gives the work done by $\vec{F}$.

Then, at any given point along C, we can compute the work done by F over a small change in $\vec{r}$, given by $\vec{dr}$, which we can assume is linear (as we take $\vec{dr}$ to be smaller and smaller). The work done along $\vec{dr}$ is then $\vec{F} \cdot \vec{dr}$, and we can then sum over all of these small pieces of $\vec{dr}$ to obtain the total work. Thus, we have that the work done by $\vec{F}$ over the path C is given by

$$W_C = \int_C \vec{F} \cdot \vec{dr}$$

and we can perform such an integral by computing

$$\vec{dr} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$

and therefore we have that

$$\int_C \vec{F} \cdot \vec{dr} = \int_C P\,dx + Q\,dy + R\,dz$$

which is an integral we know how to compute from the previous case.

Example. Let $F(x, y) = x^2y\hat{\mathbf{i}} + y^2\hat{\mathbf{j}}$. Compute the work done by F on a particle as it travels from $(0, 0)$ to $(1, 1)$ along the following curves.

- Along the curve $y = x$.
- Along the curve $y = x^2$.
- In a straight line to $(1, 0)$, then in a straight line to $(1, 1)$.
- In a straight line to $(0, 1)$, then in a straight line to $(1, 1)$.

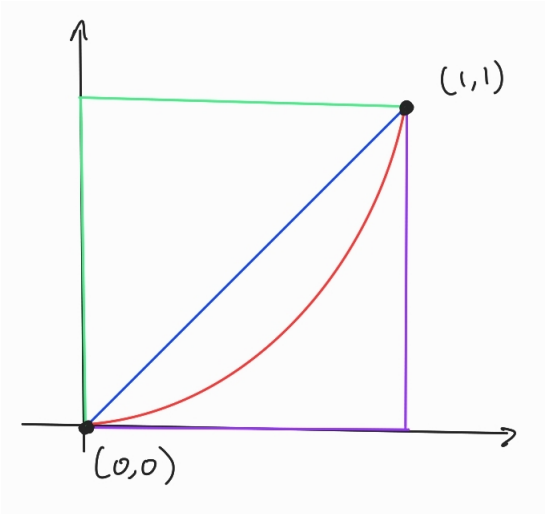


Figure 42: The four paths above, given in blue, red, purple, and green respectively.

We compute the following integrals.

- We parameterize using
$$x(t) = t \qquad y(t) = t$$
with $t \in [0, 1]$. We then have that
$$\begin{aligned}\int_C \vec{F} \cdot \vec{dr} &= \int_C x^2 y \, dx + y^2 \, dy \\ &= \int_0^1 t^3 \, dt + t^2 \, dt \\ &= \int_0^1 t^3 + t^2 \, dt \\ &= \boxed{\frac{7}{12}}\end{aligned}$$

- We parameterize using
$$x(t) = t \qquad y(t) = t^2$$
with $t \in [0, 1]$. We then have that
$$\begin{aligned}\int_C \vec{F} \cdot \vec{dr} &= \int_C x^2 y \, dx + y^2 \, dy \\ &= \int_0^1 t^4 \, dt + t^4 \, d(t^2) \\ &= \int_0^1 t^2 \, dt + (t^4)(2t) \, dt \\ &= \int_0^1 2t^5 + t^4 \, dt \\ &= \boxed{\frac{8}{15}}\end{aligned}$$

- We write down the integral
$$\int_C x^2 y \, dx + y^2 \, dy$$
and notice that along the bottom, y is the constant value 0, so $dy = 0$, and we are integrating $\int_C x^2(0) \, dx = 0$. Similarly, along the right side, $x = 1$, so $dx = 0$, and so we are integrating $\int_C y^2 \, dy$ with y going from 0 to 1. Thus, we have that
$$\int_C x^2 y \, dx + y^2 \, dy = \int_0^1 y^2 \, dy = \boxed{\frac{1}{3}}$$
Note that if the first integral had not been 0, we would have had to add the values of the two integrals.

- We again write down the integral
$$\int_C x^2 y \, dx + y^2 \, dy$$
and notice that along the left side, $x = 0$, so $dx = 0$, so we are integrating $\int_C y^2 \, dy$. Then, along the top, $y = 1$, so $dy = 0$, and we are integrating $\int_C x^2(1) \, dx$. We therefore have that
$$\int_C x^2 y \, dx + y^2 \, dy = \int_0^1 y^2 \, dy + \int_0^1 x^2 \, dx = \boxed{\frac{2}{3}}$$

A clever way to compute all four integrals at once would be to parameterize C by setting

$$x(t) = t \qquad y(t) = t^n$$

with $t \in [0, 1]$, where the curve depends on the choice of n. We would then have

$$\begin{aligned}
\int_C x^2 y\,dx + y^2\,dy &= \int_0^1 t^{n+2}\,dt + t^{2n}(nt^{n-1})\,dt \\
&= \int_0^1 t^{n+2} + nt^{3n-1}\,dt \\
&= \frac{1}{n+3} + \frac{n}{3n} \\
&= \boxed{\frac{1}{n+3} + \frac{1}{3}}
\end{aligned}$$

Note that the first two curves can be computed using $n = 1$ and $n = 2$ respectively. We can then realize the third curve as a limit when $n \to \infty$, and the fourth curve as a limit as $n \to 0$. This does indeed give the correct results!

One quick thing to note is that the integrals with respect to ds actually do not depend on the direction we traverse the curve, since we are simply summing over the arclength of the curve, but the direction does matter in the $\vec{dr}$ case. If we reverse the direction of the curve, we will negate the value of the integral.

Let's try the above example again with $\vec{F} = 2xy\hat{\mathbf{i}} + x^2\hat{\mathbf{j}}$. Using the same parameterization

$$x(t) = t \qquad y(t) = t^n$$

with $t \in [0, 1]$ we compute

$$\begin{aligned}
\int_C 2xy\,dx + x^2\,dy &= \int_0^1 2t^{n+1}\,dt + t^2(nt^{n-1})\,dt \\
&= (2+n)\int_0^1 t^{n+1}\,dt \\
&= \frac{2+n}{n+2} \\
&= \boxed{1}
\end{aligned}$$

and we notice that this value actually does not depend on n! This means regardless of which n we pick, the integral along the corresponding path gives the same value. This is in fact true for any path we pick, and is a consequence of $\vec{F}$ being conservative. We'll see why this is true in the next lecture.

Note: If you are a physics student, you may be familiar with certain work computations (gravity, for example) being independent of the path you choose. This is because the vector field associated to gravitational force is conservative (i.e., it arises as the gradient of the gravitational potential). We'll discuss this idea of being path-independent in much more detail in the upcoming lectures.

14 The Fundamental Theorem of Line Integrals

Textbook Reference: Chapter 16.3.
Planned Lecture Date(s): May 1, 2023.

Recall from single variable calculus that if we are given an integral of some function $f(x)$ on an interval $[a, b]$, we don't actually compute any Riemann sums or limits to find the value. Instead, we find an anti-derivative $F(x)$ such that $F'(x) = f(x)$, and evaluate it by computing

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

which is the statement of the fundamental theorem of calculus (or at least, one of its formulations). Since the gradient plays the role of the derivative, we should expect something similar here, and in fact this is the case! The **fundamental theorem of line integrals** tells us that for some function f and some curve C parameterized by $\vec{r}(t)$ where $a \leq t \leq b$, we have that

$$\int_C \nabla f \cdot d\vec{r} = f(r(b)) - f(r(a))$$

At first glance, this seems to be just as powerful as the fundamental theorem of calculus, but we can actually conclude more. This theorem tells us that if $\vec{F} = \nabla f$ is a conservative vector field, the integral

$$\int_C \vec{F} \cdot d\vec{r}$$

actually does not depend on the path you choose! Regardless of which choice of path C we choose to integrate over, the value of the integral depends only on the value of f at the endpoints. This is an important mantra to remember: *integrals of conservative vector fields are independent of path*. In some sense, the potential function f plays the role of an anti-derivative.

Example. Let $\vec{F} = 2xy\hat{\mathbf{i}} + (x^2 + 3y^2)\hat{\mathbf{j}}$, and let C be the curve from $(0, 0)$ to $(0, \pi)$ given by $r(\vec{t}) = (\sin(t), t)$. Compute

$$\int_C \vec{F} \cdot d\vec{r}$$

We first check that

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}$$

and thus $\vec{F}$ is possibly conservative. Integrating, we see that

$$\int P\,dx = x^2y + C_1(y) \qquad \int Q\,dy = x^2y + y^3 + C_2(x)$$

and thus we can choose a potential function

$$f(x, y) = x^2y + y^3 + C$$

where C is a constant that does not depend on either x or y. Note that in our parameterization of $r(\vec{t})$, the variable t goes from 0 to π. Then, using the fundamental theorem of line integrals, we compute

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= f(r(\pi)) - f(r(0)) \\ &= f(\pi, 0) - f(0, 0) \\ &= (0^2 \cdot \pi + \pi^3 + C) - (0 + 0 + C) \\ &= \boxed{\pi^3}\end{aligned}$$

Note that in fact the value of C did not matter, as is the case in a single-variable definite integral.

One way that we can reformulate the independence of path condition is the following: given any closed loop C (a loop must start and end at the same point), we have that

$$\int_C \nabla f \cdot \vec{dr} = 0$$

This is equivalent to the independence of path condition by the following argument.

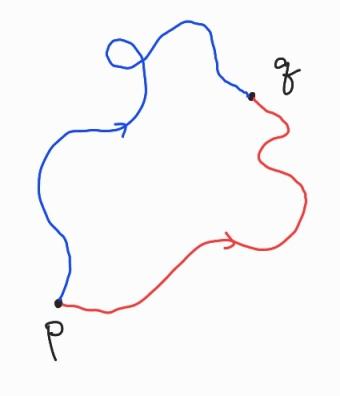


Figure 43: Two paths from p to q that we could integrate over.

In the above figure, the integral along the red path and blue path must have the same value, so if we traverse the red path followed by the blue path in reverse, our integral will be zero. We can break up any (nontrivial) loop in this manner by choosing any point q along its path and performing this decomposition. Similarly, given two paths, we can glue them together (with one in reverse) to create a loop. This shows that independence of path is indeed equivalent to all loops integrating to zero.

The fundamental theorem of line integrals tells us that if our vector field $\vec{F}$ is conservative, then our integrals are independent of path. Is the converse true? That is, if our integrals are all independent of path, does this mean we can find an f such that $\vec{F} = \nabla f$? It turns out we need a slightly stronger assumption: the domain that f is defined on must be open and connected.

A set D is **open** if no points are on the boundary of D. Examples of this include the open unit disc $\{(x, y) \,|\, x^2 + y^2 < 1\}$ and all of $\mathbb{R}^2$, but not the closed unit disc $\{(x, y) \,|\, x^2 + y^2 \leq 1\}$ and a line in $\mathbb{R}^2$. A set D is **connected** if there is a path connecting every two points in D. Examples include either the open or closed unit disc, but not two disjoint discs, or $\mathbb{R}^2$ with a line removed.

When we have this assumption on our domain D, we can actually define a potential function f as follows. Pick any point $p \in D$. Then, for any other point $q \in D$, we set

$$f(q) = \int_C \vec{F} \cdot \vec{dr}$$

for any choice of C which starts at p and ends at q. Such a path C must exist, by our assumptions on D, and the value of this integral is well-defined precisely since the integrals of $\vec{F}$ are independent of path. Checking that this indeed is a potential function for $\vec{F}$ is a bit more involved, but it does indeed work. This result lets us conclude that when our domain is open and connected, the vector fields which are conservative are precisely the ones who have integrals which are independent of path.

From last lecture, one test we could use to determine if a vector field is conservative is taking mixed partial derivatives: if $\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$, then if $\vec{F}$ is conservative, we must have that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

on our entire domain. If this condition is not satisfied, then $\vec{F}$ has no hope of being conservative. When is this condition enough to show that $\vec{F}$ is conservative? We'll illustrate this with an example.

Example. Let $\vec{F}$ be the vector field given by

$$\vec{F} = \frac{-y}{x^2+y^2}\hat{\mathbf{i}} + \frac{x}{x^2+y^2}\hat{\mathbf{j}}$$

defined on $D = \mathbb{R}^2 \setminus \{(0,0)\}$. We will show that $\vec{F}$ satisfies the partial derivative condition throughout D, but has integrals which are not independent of path.

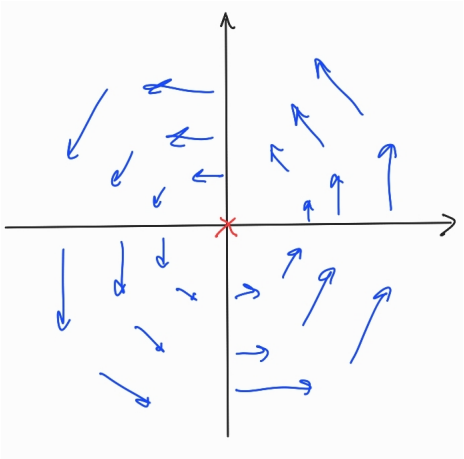


Figure 44: The vector field $\vec{F}$, colloquially known as the "vortex". Note that $\vec{F}$ is not defined at $(0,0)$.

We compute

$$\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$$

and so the partial derivative condition is indeed satisfied. However, let $p = (1,0)$, let $q = (-1,0)$, let C_1 denote the curve which follows the top half of the unit circle from p to q, and let C_2 denote the curve which follows the bottom half of the unit circle from p to q. Note that we can parameterize both C_1 and C_2 using

$$x = \cos(t) \qquad y = \sin(t)$$

but t goes from 0 to π for C_1, and t goes from 0 to $-\pi$ in C_2. We compute

$$\begin{aligned}
\int_{C_1} \vec{F} \cdot\, dr &= \int_0^{\pi} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \\
&= \int_0^{\pi} -\sin(t)(-\sin(t))\,dt + \cos(t)(\cos(t))\,dt \\
&= \int_0^{\pi} 1\,dt \\
&= \boxed{\pi}
\end{aligned}$$

Similarly, we can compute

$$\begin{aligned}
\int_{C_2} \vec{F} \cdot\, dr &= \int_0^{-\pi} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \\
&= \int_0^{-\pi} -\sin(t)(-\sin(t))\,dt + \cos(t)(\cos(t))\,dt
\end{aligned}$$

$$= \int_0^{-\pi} 1 \, dt$$
$$= \boxed{-\pi}$$

This is in fact not a sign error in our computation, as these two integrals do indeed give us different values.

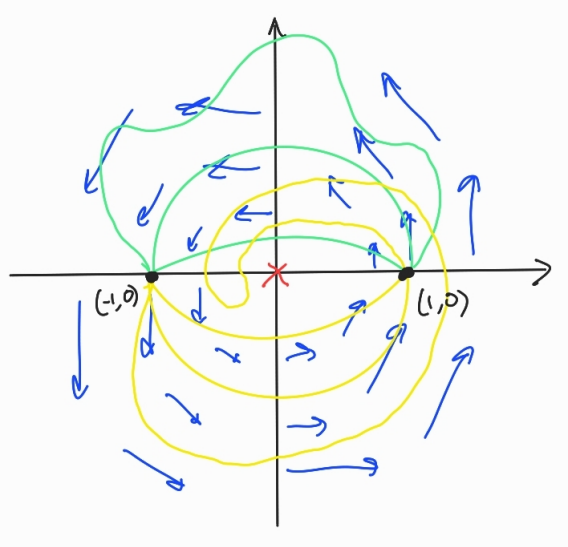


Figure 45: Different paths we can use to travel from $(1, 0)$ to $(-1, 0)$.

Note that since the partial derivative condition is satisfied everywhere in D, we can freely deform any of the green paths to another green path (and therefore those paths yield the same integral), and we can do the same for any yellow path to another yellow path, but we cannot deform a green path to a yellow path, or vice versa. The "hole" in our domain D at $(0, 0)$ prevents this deformation from happening, and the integral along our path can detect which side of the hole we have travelled through (alternatively, a loop which consists of a green curve together with a yellow curve cannot be deformed to a point, since it would have to be pulled through the "hole"). Intuitively, the green paths "follow" the flow of the vector field, and thus the integrals have positive value, whereas the yellow paths "go against" the flow of the vector field, and thus the integrals have negative value.

With this example in mind, let's return to the discussion of when the partial derivative condition is enough to conclude that our vector field is conservative. The additional assumption we need is that the domain D is **simply connected**. This means that D is connected, and furthermore, any simple (i.e. non-self intersecting) closed loops in D must contain only points of D on its interior. Intuitively, the region should not have any holes, around which a loop could wrap around.

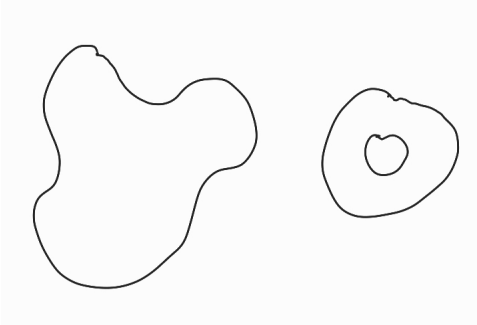


Figure 46: A simply connected region on the left, and a non-simply connected region on the right.

When D is simply connected, and furthermore

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D, then $\vec{F}$ is conservative on D. The intuition is the following: we can continuously deform our curves without affecting the value of the integral, as long as we are doing so through a region on which this partial derivative condition is satisfied. On a simply connected region, there are no holes, so every loop can be deformed by retracting it to a point, and integrating over a point will give 0. Thus, every loop has integral 0, satisfying the path-independent condition, and since D is connected, the vector field will be conservative. However, on a non-simply connected region, trying to deform a loop around a hole will get "caught" on the hole, and we cannot deform it further.

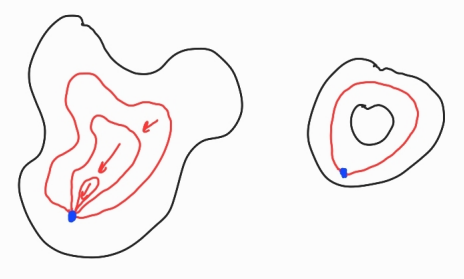


Figure 47: Deforming a curve in a simply connected region (left) versus a non-simply connected region (right).

The following discussion is not crucial to the course, but may be enlightening or just generally interesting: one may notice that in polar coordinates, we have that

$$\frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \tan(\theta)$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

and that the vector field $\vec{F}$ from the above example is precisely $\nabla\theta$, where θ is the function that returns the angle θ in polar coordinates for any (x, y) in the plane. This may worry you (and rightfully so), since the above example demonstrates that our integrals are not independent of path, and so we should not be able to produce a potential function, but θ appears to be a potential function for $\vec{F}$. What is going on here?

The answer is that θ is not a genuine function on the plane, for several reasons. The first reason is that θ is not defined at $(0, 0)$; this is easy to fix, as we can instead restrict our domain to the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$. The other is slightly harder to fix: for example, $\theta(1, 0) = 0$ by our definition of polar coordinates, but we could also just as easily say that $\theta(1, 0) = 2\pi$, or any multiple of 2π. Functions can only return one value per input, so θ is not well defined on our domain D.

The way to resolve this is to do the following: choose a ray starting at the origin, and cut the entire ray out of D. Once we have done this, D is simply connected again, and we can make a consistent choice of θ on D (we could not do this before, as on the ray we cut out, θ could have had two possible values which differed by 2π). This process is called "making a **branch cut**", where we are choosing a "branch" of θ to use.

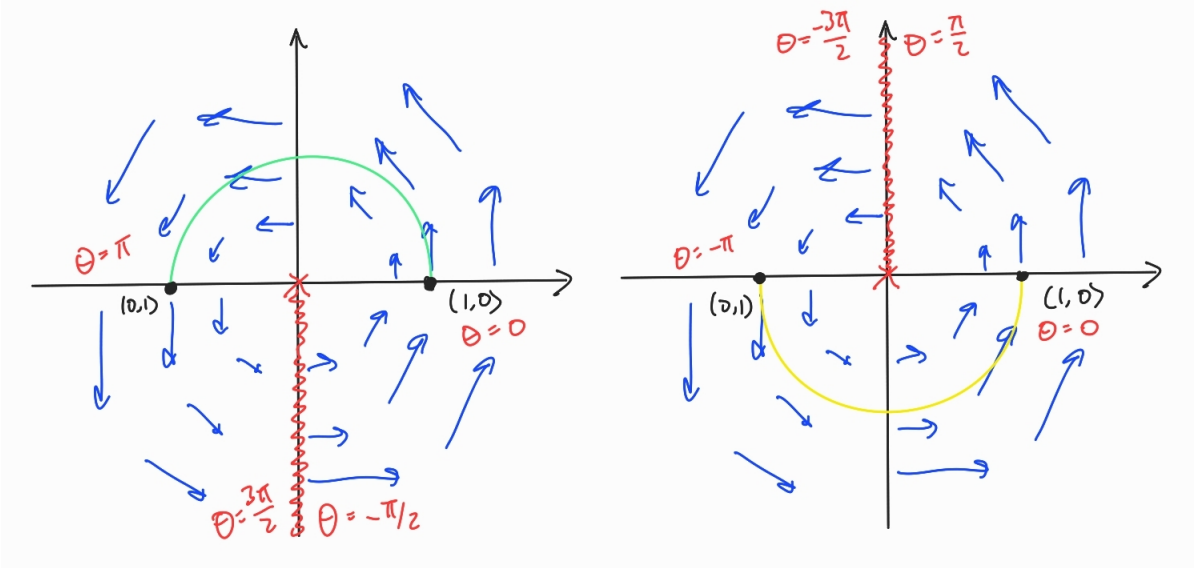


Figure 48: Making different branch cuts to accommodate the two integrals from the above example.

Once we have done this, it becomes possible to define θ consistently throughout D, so θ is an honest potential function for $\vec{F}$, and we can use the fundamental theorem of line integrals to find

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \theta(-1,0) - \theta(1,0) = \pi - 0 = \pi$$
$$\int_{C_2} \vec{F} \cdot d\vec{r} = \theta(-1,0) - \theta(1,0) = (-\pi) - 0 = -\pi$$

which agrees precisely with our above computation. It turns out that this is the "only" way that vector fields can satisfy the partial derivative condition but fail to be conservative, but I will avoid defining "only" rigorously here.

Note: Many of the terms defined in this lecture, such as open, connected, and simply connected, are not precisely the definitions used more broadly in mathematics, but will suffice for the purposes of this course. These terms are defined more precisely and investigated in a branch of math called **topology**.

15 Green's Theorem

Textbook Reference: Chapter 16.4.
Planned Lecture Date(s): May 3, 2023.

One of the key ideas of the fundamental theorem of calculus (and the fundamental theorem of line integrals) is that if we are integrating some function f over some curve C, we don't actually need to know anything about f or C aside from the values of f at the endpoints of C. The curve and the function can do whatever it wants between the endpoints of C without affecting the value of the integral. This idea that we can reduce an integral to its behavior on the boundary is the key insight behind **Green's Theorem**.

We should pause and define a simple curve - a curve is simple if it does not intersect itself. A major result known as the Jordan Curve Theorem states that any simple curve divides the plane into three regions: the curve itself, the (bounded) interior, and the (unbounded) exterior. The proof of this is remarkably lengthy for a result that seems so intuitive, but such is frequently the case in mathematics.

Let's first recall how the fundamental theorem of calculus is proved: in order to compute the integral $\int_a^b f'(x)\,dx$, we divide the interval $[a,b]$ into many small chunks $[x_i, x_{i+1}]$. Then, $f(b)-f(a)$ can be written as the sum $\sum_i f(x_{i+1})-f(x_i)$ (this sum "telescopes" to $f(b)-f(a)$), and applying the mean value theorem on this interval gives $\sum_i f'(x_i^*)\Delta x$, and taking the limit as $\Delta x \to 0$ gives the Riemann sum for the integral $\int_a^b f'(x)\,dx$. Our proof of Green's Theorem will emulate this proof.

Note that taking an integral over a zero-dimensional region, which is a collection of points, is simply a finite sum. Thus, we can think of the fundamental theorem of calculus as relating an integral over a one-dimensional region to an integral over its zero-dimensional boundary.

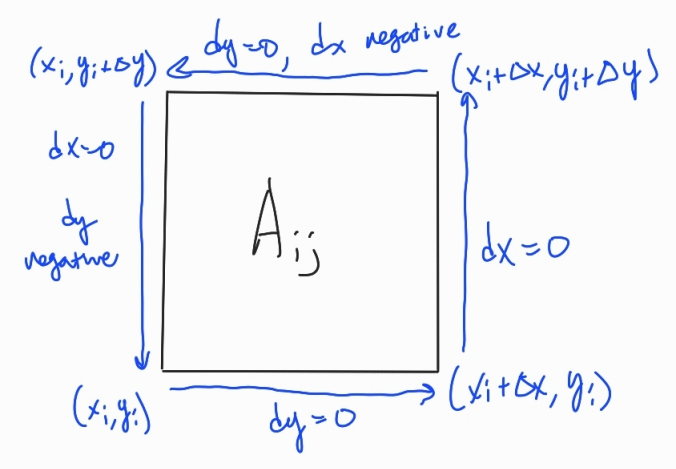


Figure 49: The contour integral around A_{ij} broken down into four separate integrals.

Suppose $\vec{F}$ is a vector field, and we would like to compute $\oint_C \vec{F}\cdot\vec{dr}$, for some simple contour C with interior D. Divide D into infinitesimally small chunks A_{ij}. In each chunk, we can perform the integral around A_{ij}, starting at the bottom left corner. We then have

$$\oint_{\partial A_{ij}} \vec{F}\cdot\vec{dr} = \int_{x_i}^{x_i+\Delta x} P(x', y_i)\,dx' + \int_{y_i}^{y_i+\Delta y} Q(x_i+\Delta x, y')\,dy'$$

$$
\begin{aligned}
&\quad - \int_{x_i}^{x_i+\Delta x} P(x', y_i + \Delta y)\, dx' \\
&\quad - \int_{y_i}^{y_i+\Delta y} Q(x_i, y')\, dy' \\
&= \int_{y_i}^{y_i+\Delta y} Q(x_i + \Delta x, y') - Q(x_i, y')\, dy' \\
&\quad - \int_{x_i}^{x_i+\Delta x} P(x', y_i + \Delta y) - P(x', y_i)\, dx'
\end{aligned}
$$

We now apply the Mean Value Theorem to find a value x^* and y^* with $(x^*, y^*) \in A_{ij}$ such that the partial derivatives agree with the overall change in Q and P respectively.

$$
= \int_{y_i}^{y_i+\Delta y} \frac{\partial Q}{\partial x}(x^*)\Delta x\, dy' - \int_{x_i}^{x_i+\Delta x} \frac{\partial P}{\partial y})(y^*)\Delta y\, dx'
$$

Taking Δx and Δy to be infinitesimally small, we can approximate the integrand as constant on this interval, so we obtain

$$
= \frac{\partial Q}{\partial x}(x^*)\Delta x \Delta y - \frac{\partial P}{\partial y}(y^*)\Delta x \Delta y
$$

The quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ can be thought of as the "circulation" of $\vec{F}$ around the small chunk A_{ij}. We then claim that

$$
\oint_C \vec{F} \cdot \vec{dr} = \sum_{i,j} \oint_{A_{i,j}} \vec{F} \cdot \vec{dr} \to \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA
$$

as the contributions from adjacent $A_{i,j}$ regions will cancel, as given by the diagram below.

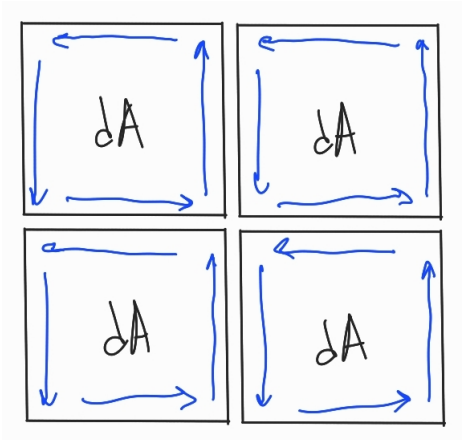


Figure 50: Circulation from adjacent regions will cancel, leaving only the circulation on the boundary.

Since the circulation of adjacent regions will go in opposite directions, the net contribution will be zero - unless the chunk dA does not have an adjacent region in some direction, in which case it is on the boundary. Thus, the contour integral around all of D reduces to the contour integral of each $A_{i,j}$, and taking a limit as each $A_{i,j}$ gets smaller allows us to instead integrate the value $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. It turns out that this quantity is example of the **curl** of a vector field (in two dimensions), which we'll talk about in the next lecture.

Green's Theorem states that for a simple closed (i.e. loop) curve C, which has interior D, we have the following relationship:

$$
\oint_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA
$$

Note that this relates an area integral, throughout the entire region D, to a path integral (of the appropriate quantity) around the boundary. In this manner, you should think of Green's Theorem as a higher dimensional analogue of the fundamental theorem of calculus.

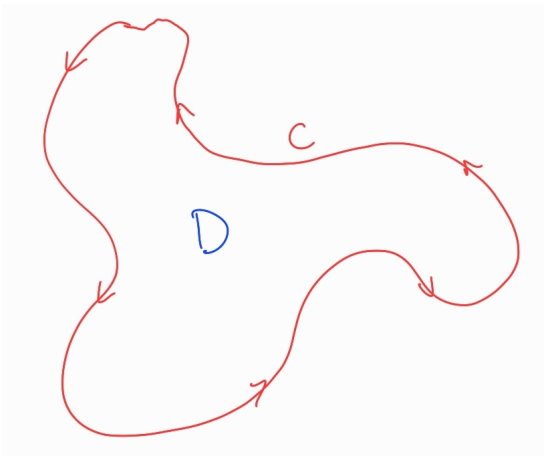


Figure 51: A closed path C (oriented counterclockwise) and the region it bounds.

A slight ambiguity does need to be resolved here: remember that if we reverse the direction of a curve C, the integral will change sign, but both C and $-C$, the reversed curve, define the same interior D. The choice we make is that as we traverse C, the interior of C should be to the left. This amounts to (generally) moving around D counterclockwise, and this choice has to do with the previous discussion of orientation.

Note: The symbol $\oint$ refers to a path integral over a closed loop. We will also frequently use ∂D to denote the boundary of D, so in this case, $\partial D = C$.

One of the first things you may notice is that if a vector field $\vec{F}$ satisfies the partial derivative condition

$$\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} \qquad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

throughout the domain D, then if we try to integrate $\int_C \vec{F} \cdot \vec{dr}$ for some loop C, by Green's Theorem, this is the same as integrating 0 over the area D, and thus our closed loops integrate to 0 as desired. Furthermore, if a vector field is not conservative, Green's Theorem lets us "take out" any part that is conservative, and not have to worry about it (because its contributions will cancel in the difference of the partial derivatives).

Example. Compute

$$\oint_C \left(3y - e^{\sin(x)}\right)\, dx + \left(7x + \sqrt{y^4+1}\right)\, dy$$

where C is the closed loop $x^2 + y^2 = 9$.

We apply Green's Theorem on the domain $D = \{(x, y) \,|\, x^2 + y^2 \leq 9\}$. We have that

$$\begin{aligned}
\oint_C (3y - e^{\sin(x)})\, dx + (7x + \sqrt{y^4+1})\, dy &= \iint_D \frac{\partial}{\partial x}\left(7x + \sqrt{y^4+1}\right) - \frac{\partial}{\partial y}\left(3y - e^{\sin(x)}\right)\, dA \\
&= \iint_D 7 - 3\, dA \\
&= 4\iint_D 1\, dA \\
&= \boxed{36\pi}
\end{aligned}$$

This saved us a lot of work, as computing these as path integrals would be incredibly difficult.

We could also use Green's Theorem in the reverse direction. Suppose that we have a region D which is the interior of some curve C, and we can parameterize C easily. Then, if we find a vector field $\vec{F}$ such that

$$\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

then we can compute the area of D using Green's Theorem by computing

$$\text{Area} = \iint_D 1\, dA = \oint_{\partial D} \vec{F} \cdot\, dr$$

and instead evaluating the contour integral. Some choices we can use include

- $\vec{F} = x\hat{\mathbf{j}}$
- $\vec{F} = -y\hat{\mathbf{i}}$
- $\vec{F} = \frac{1}{2}(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})$

and integrating any of these by finding $\oint_{\partial D} \vec{F} \cdot\, dr$ will give us the area.

Example. Let C be the curve defined by $\vec{r(t)}$, where

$$\vec{r(t)} = \begin{pmatrix} 4\cos(t) + \sin(4t) & 9\sin(t) + \cos(3t) \end{pmatrix}$$

and t varies from 0 to 2π. Find the area of the region enclosed by C.

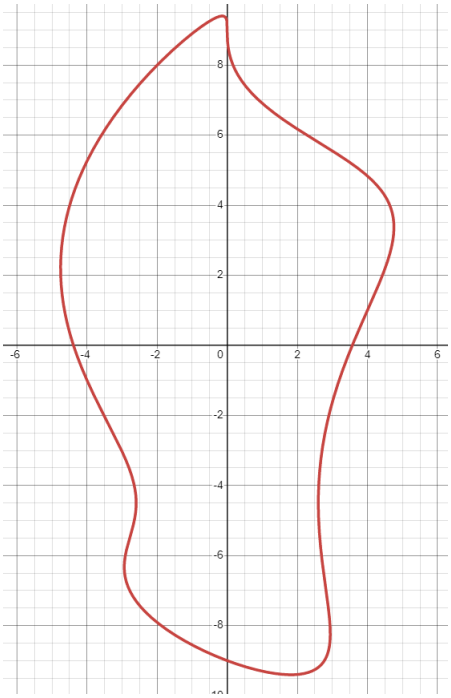


Figure 52: The region bounded by C.

We first compute

$$dx = (-4\sin(t) + 4\cos(4t))\, dt \qquad dy = (9\cos(t) - 3\sin(3t))\, dt$$

We then have that

$$\text{Area} = \iint_D 1\, dA$$

$$
\begin{aligned}
&= \oint x\,dy \\
&= \int_0^{2\pi} (4\cos(t) + \sin(4t))(9\cos(t) - 3\sin(3t))\,dt \\
&= \int_0^{2\pi} 36\cos^2(t) + 9\cos(t)\sin(4t) - 12\cos(t)\sin(3t) - 3\sin(3t)\sin(4t)\,dt
\end{aligned}
$$

Since $\sin(kt)$ and $\cos(kt)$ are periodic with periods dividing 2π, we can instead integrate over $-\pi$ to π, on which $\sin(kt)$ is odd and $\cos(kt)$ is even. Thus, the second and third terms of this integral product to odd functions of t integrated over a symmetric interval, and thus are 0. The final term can be rewritten as $-\frac{3}{2}(\cos(t) - \cos(7t))$, and these both have period dividing 2π, so will also integrate to zero. We therefore have

$$
\begin{aligned}
&= 36\int_0^{2\pi} \cos^2(t)\,dt \\
&= \boxed{36\pi}
\end{aligned}
$$

where in this final integral, we observe that

$$
I = \int_0^{2\pi} \cos^2(t)\,dt = \int_0^{2\pi} \sin^2(t)\,dt
$$

as the sine and cosine functions are just shifted versions of each other, and we are integrating over an entire period. Furthermore, we have that

$$
2I = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi
$$

so we conclude that $I = \pi$.

We can also get a little bit creative with Green's Theorem. Recall from our previous example where

$$
\vec{F} = \frac{-y}{x^2+y^2}\hat{\mathbf{i}} + \frac{x}{x^2+y^2}\hat{\mathbf{j}}
$$

that if we integrate over the hole at $(0,0)$, we obtain π, and if we integrate under the hole, we get $-\pi$. Thus, if we integrate all the way around, we should get $\pi - (-\pi) = 2\pi$. Let's suppose we instead choose a smaller contour going around the hole at $(0,0)$.

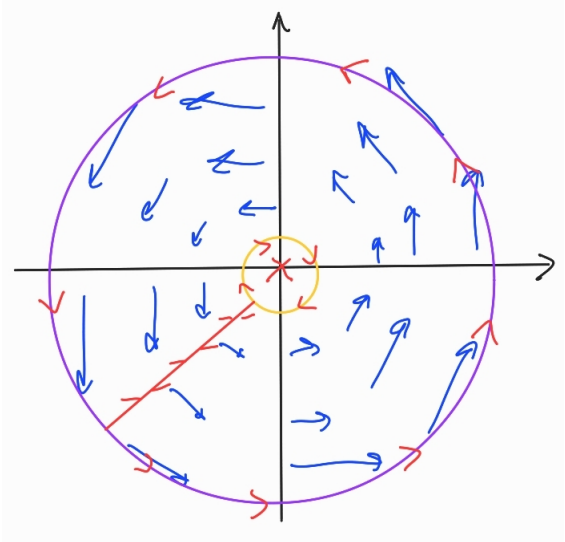


Figure 53: The purple curve, for which we have computed the integral, and the orange curve, a smaller contour for which we have not computed the integral.

Let C_1 denote the purple contour and let C_2 denote the orange contour. We can then choose our contour C to follow the purple curve in the counterclockwise direction, then cut across our region to the orange curve, which we will follow in the clockwise direction. After going around the orange curve, we will cut back across the region to the purple contour, and follow it the rest of the way. Let C_3 denote the cut across the region. Then, we have that

$$\oint_C \vec{F} \cdot \vec{dr} = \int_{C_1} \vec{F} \cdot \vec{dr} + \int_{C_3} \vec{F} \cdot \vec{dr} - \int_{C_2} \vec{F} \cdot \vec{dr} - \int_{C_3} \vec{F} \cdot \vec{dr}$$

where the minus signs are chosen because of the orientation on each curve. This then simplifies to

$$\oint_C \vec{F} \cdot \vec{dr} = \int_{C_1} \vec{F} \cdot \vec{dr} - \int_{C_2} \vec{F} \cdot \vec{dr}$$

but additionally, by Green's Theorem, since we showed that $\vec{F}$ satisfies the partial derivative condition everywhere away from $(0, 0)$, we have that

$$\oint_C \vec{F} \cdot \vec{dr} = \iint_D 0 \, dA = 0$$

and therefore we conclude that

$$\int_{C_1} \vec{F} \cdot \vec{dr} = \int_{C_2} \vec{F} \cdot \vec{dr}$$

which shows that any contour around the hole at $(0, 0)$ integrates to the same value, 2π.

16 Curl and Divergence

Textbook Reference: Chapter 16.5.
Planned Lecture Date(s): May 5, 2023.

At the end of last lecture, we defined the quantity

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

associated to a vector field $P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ as a measure of its circulation. This is in fact a specialization of the following operation: given a vector field

$$\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$$

we can define the **curl** of $\vec{F}$ as $\nabla \times \vec{F}$, where we treat ∇ as the vector with partial derivative operators as entries, and take the cross product with $\vec{F}$. Algebraically, this looks like

$$\begin{aligned}\text{curl}(F) = \nabla \times \vec{F} &= \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}\end{aligned}$$

and we notice that if $R = 0$, and also P and Q do not depend on z, we recover

$$\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

which is precisely the expression we had before, except now it is a vector in the $\hat{\mathbf{k}}$ direction. Note that the curl of a vector field is another vector field.

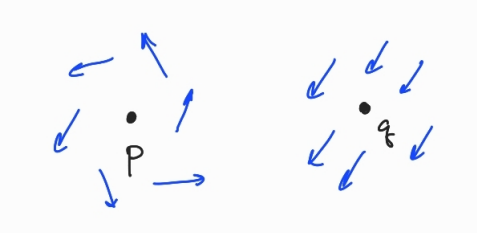


Figure 54: A vector field which exhibits nonzero curl at p, and zero curl at q.

Geometrically, the curl represents the amount that a vector field rotates around any point. You can think of the vector field $\vec{F}$ as the flow of a liquid, so if you place a small paddle wheel into the liquid at some point, the curl represents how much the paddle wheel will spin. For example, in the case of the vortex vector field, even though the vector field is globally "rotating", if we put a paddle wheel down at any point (aside from $(0, 0)$), near that point all of the vectors are pointing in the same direction.

A vector field for which $\nabla \times \vec{F} = 0$ is called **irrotational**. Physics students may also recognize that the curl vector points in the direction of the axis of rotation, according to the right-hand rule.

We can also define another quantity, the **divergence** of $\vec{F}$, as $\nabla \cdot \vec{F}$. Algebraically, we write

$$\text{Div}(\vec{F}) = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) \cdot (P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

and note that even when we are in the two-dimensional case and $R = 0$, this quantity still makes sense (although it will not be of too much use to us). Note that the divergence of a vector field is not a vector (field), but instead a scalar quantity (which can depend on x, y, and z).

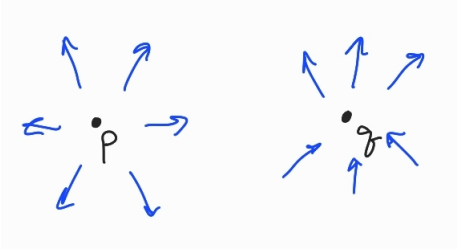


Figure 55: A vector field which exhibits nonzero divergence at p, and zero divergence at q.

Geometrically, the divergence represents how much the vector field is expanding at a certain point. In the example where $\vec{F}$ represents the flow of a fluid, a point with nonzero divergence would be a point from which the fluid is entering the system. A vector field for which $\nabla \cdot \vec{F} = 0$ is called **incompressible**.

Example. For the following vector fields:

- $\vec{F} = xz\hat{\mathbf{i}} + xyz\hat{\mathbf{j}} - y^2\hat{\mathbf{k}}$
- $\vec{G} = (-2y - xy)\hat{\mathbf{i}} + x\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$

compute the curl and divergence.

- We compute

$$\begin{aligned}\nabla \times \vec{F} &= \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{pmatrix} \\ &= (-2y - xy)\hat{\mathbf{i}} + (x - 0)\hat{\mathbf{j}} + (yz - 0)\hat{\mathbf{k}} \\ &= \boxed{(-2y - xy)\hat{\mathbf{i}} + x\hat{\mathbf{j}} + yz\hat{\mathbf{k}}}\end{aligned}$$

  and similarly

$$\begin{aligned}\nabla \cdot \vec{F} &= \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) \cdot (xz\hat{\mathbf{i}} + xyz\hat{\mathbf{j}} - y^2\hat{\mathbf{k}}) \\ &= z + xz + 0 \\ &= \boxed{z(x+1)}\end{aligned}$$

- We compute

$$\begin{aligned}\nabla \times \vec{F} &= \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y - xy & x & yz \end{pmatrix} \\ &= z\hat{\mathbf{i}} + (0)\hat{\mathbf{j}} + (1 - (-2 - x))\hat{\mathbf{k}} \\ &= \boxed{z\hat{\mathbf{i}} + (3 + x)\hat{\mathbf{k}}}\end{aligned}$$

  and similarly

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right) \cdot (-2y - xy)\hat{\mathbf{i}} + x\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$$

$$= -y + 0 + y$$
$$= \boxed{0}$$

One may notice that in the above example, $\vec{G} = \nabla \times \vec{F}$, and furthermore, $\nabla \cdot \vec{F} = 0$. This is not a coincidence! In general, we have the following identities, which you can check algebraically (and they ultimately come down to Clairaut's Theorem on the symmetry of second partials):

- $\nabla \times (\nabla f) = 0$
- $\nabla \cdot (\nabla \times \vec{F}) = 0$

This can be summarized in the following diagram, where the composition of any two of the operations will give zero.

$$\{\text{Functions}\} \xrightarrow{f \mapsto \nabla f} \{\text{Vector Fields}\} \xrightarrow{\vec{F} \mapsto \nabla \times \vec{F}} \{\text{Vector Fields}\} \xrightarrow{\vec{F} \mapsto \nabla \cdot \vec{F}} \{\text{Functions}\}$$

This gives us a test to see when functions of three variables are conservative: if $\nabla \times \vec{F} \neq 0$, then $\vec{F}$ cannot be conservative. We had previously stated this in terms of the individual quantities, but using the language of vector calculus, we can bundle this claim up into a nice, neat statement. We can also use this to detect if a vector field is the curl of another vector field, as if $\nabla \cdot \vec{F} \neq 0$, then $\vec{F}$ cannot be the curl of some other vector field.

One may recall that in the two-dimensional case, on a simply connected domain, checking the partial derivative condition was not only necessary, but a sufficient condition for $\vec{F}$ to be conservative. This can be extended to three dimensions, as $\vec{F}$ is conservative if and only if $\nabla \times \vec{F} = 0$. Furthermore, on a domain without holes (this can be formalized mathematically, but not within the scope of this course), a vector field $\vec{F}$ is the curl of another vector field if and only if $\nabla \cdot \vec{F} = 0$. We can summarize this in the above diagram by remembering that (on a domain without holes) if any object is mapped to 0 along one of the operations, then it must have come from an object along the previous arrow.

Using our newly defined language, we can rephrase Green's Theorem in a manner that will be more useful to us going forward. Given some vector field

$$\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$$

and some simple curve C with interior D, we see that Green's Theorem states that

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\nabla \times \vec{F}) \cdot \hat{\mathbf{k}} \, dA$$

where we recall that when $\vec{F}$ is only in two dimensions, we have that

$$\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{\mathbf{k}}$$

and so taking a dot product with $\hat{\mathbf{k}}$ gives precisely the quantity we want. Moving forward, it will actually be helpful to think of $\hat{\mathbf{k}} \, dA$ as one object, which keeps track of both a (normal) vector and a small chunk of area - we will see what this means in a few lectures.

Note: The fact that the composition of any two operations gives zero in the sequence of ∇-related operators is actually incredibly useful when the study of vector calculus is generalized further, but this is beyond the scope of this course. If you are interested, feel free to investigate **differential forms** or to take a course on **differential geometry**.

17 Parametric Surfaces

Textbook Reference: Chapter 16.6.
Planned Lecture Date(s): May 8, 2023.

Throughout our discussion of line integrals, we've parameterized our curves using a variable $t \in [a, b]$, and thought of t as a time parameter: as t increases, we travel along our curve. Thus, if we allow t to travel along the entirety of $[a, b]$, our coordinate functions will trace out the entirety of our curve. For simplicity, we'll often take $[a, b] = [0, 1]$.

In other words, our position function

$$\vec{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

defines a function which takes in a single variable t, and outputs a point in three-dimensional space. As the variable t varies, it defines a curve. Intuitively, we think of a curve as one-dimensional, and this is formalized in the fact that we can parameterize it using one variable. The fact that we are able to parameterize our curve is what allows us to do calculus on it.

What happens if we instead allow two variables? That is, our position function $r(\vec{t})$ is instead given by

$$\vec{r}(u, v) = x(u, v)\hat{\mathbf{i}} + y(u, v)\hat{\mathbf{j}} + z(u, v)\hat{\mathbf{k}}$$

for some real variables u and v. Intuitively, this gives us two different directions to move in: we can increase or decrease u, and we can similarly increase or decrease v. Thus, the resulting region of space that $\vec{r}(u, v)$ defines should be two-dimensional, which we think of as a surface.

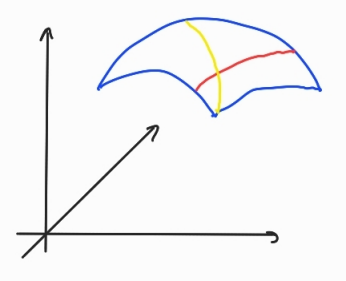


Figure 56: A surface parameterized by u and v, with a line of constant u and v represented by a red and yellow line, respectively.

The variables u and v are our **parameters**: as we vary u and v over some values of our choosing, the function $\vec{r}(u, v)$ traces out a surface, and near any point, changing u and v by a small amount corresponds to the directions that we can travel along our surface.

You might be thinking: why would we go through all of this effort to parameterize surfaces? The answer is that this generalizes many of our existing notions of surfaces, but allows us to do more with it.

- For example, let $z = f(x, y)$ define a surface. Then, we can simply use x and y as parameters! As we change x and y, the variable z will change according to the rule $z = f(x, y)$, and so this defines a two-dimensional surface lying over the xy-plane.
- Let E be the sphere of radius R (without the interior). This is a two-dimensional surface, so we should be able to parameterize it with two variables. In fact, we already know how to do this using spherical coordinates! Fix $\rho = R$, and let θ and φ vary between $[0, 2\pi]$ and $[0, \pi]$ respectively. This will trace out the entire sphere of radius R, and we can use θ and φ as parameters.

- However, parametric surfaces allow us to define even more surfaces which we previously could not define. For example, the (unfilled) torus with "ring" radius R and "tube" radius r can be given by the parametric equations

$$x(\theta,\varphi) = (R + r\cos(\varphi))\cos(\theta) \qquad y(\theta,\varphi) = (R + r\cos(\varphi))\sin(\theta) \qquad z(\theta,\varphi) = r\sin(\varphi)$$

with $\theta, \varphi \in [0, 2\pi]$. The way to visualize these coordinates is to think of θ as the coordinate which takes you around the torus, and at a fixed θ, the cross-section of the torus is a circle, which is parameterized by φ.

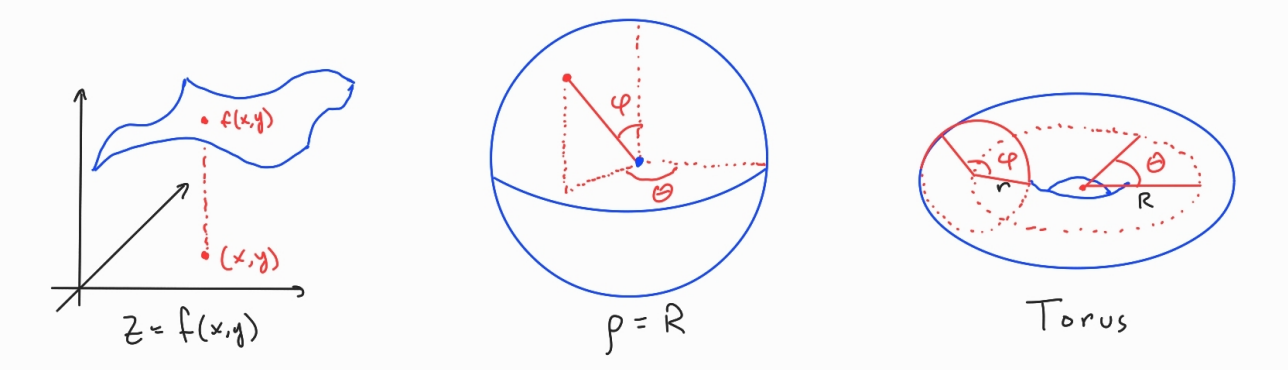


Figure 57: The three surfaces parameterized above.

In many ways, choosing a set of parameters is like a change of coordinates: the surfaces that we want to parameterize come with a natural choice of coordinates (x, y, z), but these coordinates satisfy some constraint which defines the surface. This constraint essentially makes one of the variables redundant, so we can parameterize with two variables instead, and choosing some appropriate choice of (u, v) gives us a set of variables to work with.

Once we are able to parameterize our surfaces, we are able to do calculus on them! For example, if we want to find the **normal vector** to a parametric surface S at a point (u, v), we can compute

$$\vec{r}_{\vec{u}} = \frac{\partial x}{\partial u}\hat{\mathbf{i}} + \frac{\partial y}{\partial u}\hat{\mathbf{j}} + \frac{\partial z}{\partial u}\hat{\mathbf{k}} \qquad \vec{r}_{\vec{v}} = \frac{\partial x}{\partial v}\hat{\mathbf{i}} + \frac{\partial y}{\partial v}\hat{\mathbf{j}} + \frac{\partial z}{\partial v}\hat{\mathbf{k}}$$

as our "directional derivatives" in u and v, and thus their cross product will give a normal vector to the surface at every point.

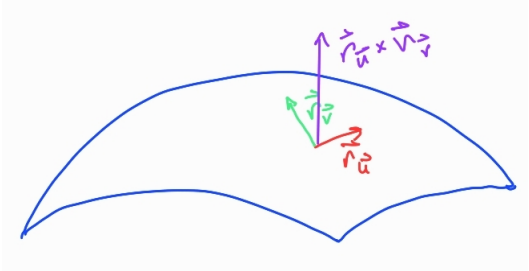


Figure 58: The two vectors $\vec{r}_{\vec{u}}$ and $\vec{r}_{\vec{v}}$ are "directional derivatives" in u and v, and thus want to stay in the tangent plane to the surface, so their cross product will be normal to the surface.

We can then extract the tangent plane from this information, as if

$$\vec{r}_{\vec{u}} \times \vec{r}_{\vec{v}} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$$

then just as before, we can write

$$P(x_0, y_0, z_0)(x - x_0) + Q(x_0, y_0, z_0)(y - y_0) + R(x_0, y_0, z_0)(z - z_0) = 0$$

for equation of the tangent plane at a point (x_0, y_0, z_0). The vector $\vec{r}_{\vec{u}} \times \vec{r}_{\vec{v}}$ is called the **surface normal** vector, and will be very useful to us.

Example. Find a normal vector and the equation of a tangent plane to a sphere of radius R.

From spherical coordinates, we know that we can choose the parameterization

$$x = R\cos(\theta)\sin(\varphi) \qquad y = R\sin(\theta)\sin(\varphi) \qquad z = R\cos(\varphi)$$

where $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$, and R is fixed (note that ρ does not occur in our parameterization). We then compute

$$\begin{aligned}
\vec{r}_{\vec{\theta}} &= -R\sin(\theta)\sin(\varphi)\hat{\mathbf{i}} + R\cos(\theta)\sin(\varphi)\hat{\mathbf{j}} \\
\vec{r}_{\vec{\varphi}} &= R\cos(\theta)\cos(\varphi)\hat{\mathbf{i}} + R\sin(\theta)\cos(\varphi)\hat{\mathbf{j}} - R\sin(\varphi)\hat{\mathbf{k}} \\
\vec{r}_{\vec{\theta}} \times \vec{r}_{\vec{\varphi}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -R\sin(\theta)\sin(\varphi) & R\cos(\theta)\sin(\varphi) & 0 \\ R\cos(\theta)\cos(\varphi) & R\sin(\theta)\cos(\varphi) & -R\sin(\varphi) \end{vmatrix} \\
&= (-R^2\cos(\theta)\sin^2(\varphi))\hat{\mathbf{i}} - (R^2\sin(\theta)\sin^2(\varphi))\hat{\mathbf{j}} \\
&\quad + (-R^2\sin^2(\theta)\sin(\varphi)\cos(\varphi) - R^2\cos^2(\theta)\sin(\varphi)\cos(\varphi))\hat{\mathbf{k}} \\
&= (-R^2\cos(\theta)\sin^2(\varphi))\hat{\mathbf{i}} - (R^2\sin(\theta)\sin^2(\varphi))\hat{\mathbf{j}} + (-R^2\sin(\varphi)\cos(\varphi))\hat{\mathbf{k}} \\
&= -R\sin(\varphi)\Big(R\cos(\theta)\sin(\varphi)\hat{\mathbf{i}} + R\sin(\theta)\sin(\varphi)\hat{\mathbf{j}} + R\cos(\varphi)\hat{\mathbf{k}}\Big)
\end{aligned}$$

and we notice that (up to the factor of $-R\sin(\varphi)$), the vector in parentheses is simply

$$x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

which we can verify from our intuition is indeed the normal vector to a sphere!

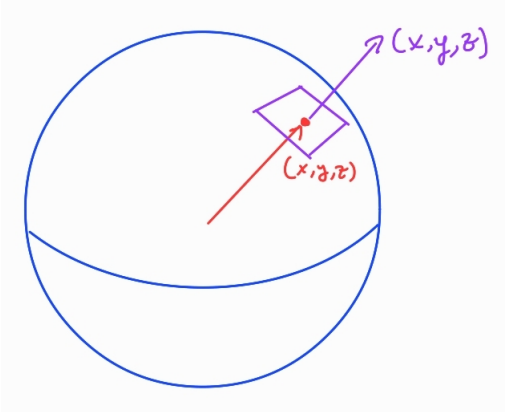


Figure 59: The normal vector to a sphere at any point is in the same direction as the position vector.

There are a few interesting things to note here: the first is that the coefficient attached to this cross product is, in fact, negative. This corresponds to the fact that the order we chose to take the cross product (i.e., $\vec{r}_{\vec{\theta}} \times \vec{r}_{\vec{\varphi}}$ rather than $\vec{r}_{\vec{\varphi}} \times \vec{r}_{\vec{\theta}}$) gave us the inward-pointing normal vector, rather than the outward-pointing normal vector (an indication that our coordinate system is left-handed). Generally, we will choose the outward-pointing normal vector, but this is a choice of orientation that we make on our surface.

The other interesting thing to note is that as φ approaches either 0 or π, the normal vector actually goes to $\vec{0}$. This is also to be expected, since our coordinates run into some issues at these points: in particular, θ fails to make sense, and so $d\theta$ also fails to make sense. We don't really have a good solution for this, but since it happens only at two points, most of our calculus allows us to ignore this.

However, remember from our discussion of surface area that this cross product actually carries more information! We've thus far only used the direction of the surface normal vector, but the magnitude is also meaningful to us. The magnitude of the surface normal represents the area of the tangent plane above a small chunk of area, and so if we'd like to find the surface area of a parametric surface, we can integrate the magnitude of the surface normal over our domain. This is given by the formula

$$\text{SA}(S) = \iint_D |\vec{r}_{\vec{u}} \times \vec{r}_{\vec{v}}| \, dA$$

where D represents a domain over which (u, v) range. Note that in the special case where our surface is the graph of a function, we can parameterize using x and y, and so we have

$$x = x \qquad y = y \qquad z = f(x, y)$$

and computing

$$\begin{aligned}
\vec{r}_{\vec{x}} &= \hat{\mathbf{i}} + \frac{\partial f}{\partial x}\hat{\mathbf{k}} \\
\vec{r}_{\vec{y}} &= \hat{\mathbf{j}} + \frac{\partial f}{\partial y}\hat{\mathbf{k}} \\
\vec{r}_{\vec{x}} \times \vec{r}_{\vec{y}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} \\
&= -\frac{\partial f}{\partial x}\hat{\mathbf{i}} - \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}
\end{aligned}$$

This vector is worth remembering, as it is the surface normal to the surface of a graph $z = f(x, y)$. Note then that taking the magnitude of this vector gives

$$|\vec{r}_{\vec{x}} \times \vec{r}_{\vec{y}}| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

which is precisely the formula we previously had. However, we now have the ability to compute surface areas of surfaces which are not of the form $z = f(x, y)$, using our new tools.

We will often write the surface normal together with dA to form $d\vec{S}$, often called the **surface element**, as it represents both a small chunk of surface area as well as the direction of the normal to the surface.

18 Surface Integrals

Textbook Reference: Chapter 16.7.
Planned Lecture Date(s): May 15, 2023.

We went through all the trouble of defining parametric surfaces in order to make it easier for us to do calculus on these surfaces, and now it's time to reap the rewards of our labor. Let's recall that for a surface parameterized by u and v, we defined the surface normal to be

$$d\vec{S} = \vec{r}_u \times \vec{r}_v \, dA$$

where $\vec{r}_u$ and $\vec{r}_v$ are the partial derivatives of $\vec{r}(u, v)$ with respect to u and v. We think of $d\vec{S}$ as an vector which keeps track of two things: the direction is normal to the surface at every point, and the magnitude represents the area of the tangent plane lying above a small chunk of area.

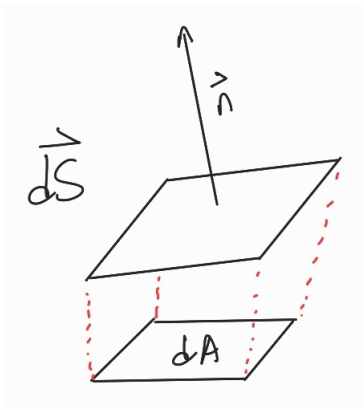


Figure 60: The information that $d\vec{S}$ keeps track of.

Another way of writing this is

$$d\vec{S} = \vec{n} \, dA = \hat{\mathbf{n}} \, dS$$

where $\vec{n}$ is the surface normal vector and $\hat{\mathbf{n}}$ is $\vec{n}$ normalized to be a unit vector. If we'd simply like to integrate some function f over the surface, without worrying about the normal direction, we can do so by integrating over small chunks dS of the surface, where $dS = |d\vec{S}| = |\vec{n}| \, dA$. Our formula then becomes

$$\iint_S f(x, y, z) \, dS = \iint_D f(r(u, v)) |\vec{n}| \, dA$$

where D is some domain in (u, v). One such example is when we'd like to find the surface area, as we would integrate the constant function 1 over the surface.

Example. Find the surface area of the torus with "ring" radius R and "tube" radius r, given by the parametric equations

$$x(\theta, \varphi) = (R + r\cos(\varphi))\cos(\theta) \qquad y(\theta, \varphi) = (R + r\cos(\varphi))\sin(\theta) \qquad z(\theta, \varphi) = r\sin(\varphi)$$

where θ and φ both vary from 0 to 2π.

We first compute the surface normal, as we have

$$\vec{r}_\theta = -(R + r\cos(\varphi))\sin(\theta)\hat{\mathbf{i}} + (R + r\cos(\varphi))\cos(\theta)\hat{\mathbf{j}}$$
$$\vec{r}_\varphi = -r\sin(\varphi)\cos(\theta)\hat{\mathbf{i}} - r\sin(\varphi)\sin(\theta)\hat{\mathbf{j}} + r\cos(\varphi)\hat{\mathbf{k}}$$

$$\begin{aligned}
\vec{r}_{\vec{\theta}} \times \vec{r}_{\vec{\varphi}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -(R + r\cos(\varphi))\sin(\theta) & (R + r\cos(\varphi))\cos(\theta) & 0 \\ -r\sin(\varphi)\cos(\theta) & -r\sin(\varphi)\sin(\theta) & r\cos(\varphi) \end{vmatrix} \\
&= r(R + r\cos(\varphi))\cos(\theta)\cos(\varphi)\hat{\mathbf{i}} + r(R + r\cos(\varphi))\sin(\theta)\cos(\varphi) \\
&\quad + (r(R + r\cos(\varphi))\sin(\varphi)\sin^2(\theta) + r(R + r\cos(\varphi))\sin(\varphi)\cos^2(\theta))\hat{\mathbf{k}} \\
&= r(R + r\cos(\varphi))\cos(\theta)\cos(\varphi)\hat{\mathbf{i}} + r(R + r\cos(\varphi))\sin(\theta)\cos(\varphi) \\
&\quad + r(R + r\cos(\varphi))\sin(\varphi)\hat{\mathbf{k}}
\end{aligned}$$

We then compute that

$$\begin{aligned}
|\vec{r}_{\vec{\theta}} \times \vec{r}_{\vec{\varphi}}| &= r(R + r\cos(\varphi))\sqrt{\cos^2(\theta)\cos^2(\varphi) + \sin^2(\theta)\cos^2(\varphi) + \sin^2(\varphi)} \\
&= r(R + r\cos(\varphi))\sqrt{\cos^2(\varphi) + \sin^2(\varphi)} \\
&= r(R + r\cos(\varphi))
\end{aligned}$$

and so we have that

$$dS = r(R + r\cos(\varphi))\, dA$$

Let $D = [0, 2\pi] \times [0, 2\pi]$ be a rectangle in the (θ, φ)-domain. We then compute

$$\begin{aligned}
\text{Surface Area} &= \iint_S 1\, dS \\
&= \iint_D r(R + r\cos(\varphi))\, dA \\
&= r\int_0^{2\pi}\int_0^{2\pi} R + r\cos(\varphi)\, d\theta\, d\varphi \\
&= 2\pi r\int_0^{2\pi} R + r\cos(\varphi)\, d\varphi \\
&= (2\pi r)(2\pi R + 0) \\
&= \boxed{4\pi^2 Rr}
\end{aligned}$$

which gives us a formula for the surface area of a torus.

However, this integral completely ignored the normal direction that comes with $d\vec{S}$! Just like the line integral case, we can actually do more with surface integrals. The motivation comes from the notion of **flux**, from physics. Given a surface S and a vector field $\vec{F}$, we would like to compute how much $\vec{F}$ "goes through" S.

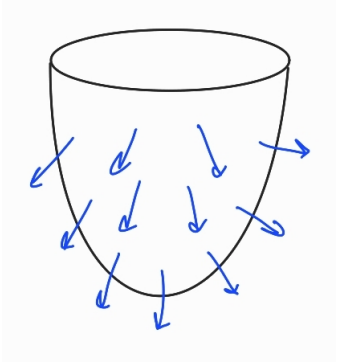


Figure 61: Flux measures the amount that the blue vector field goes through the paraboloid.

A motivating example of a flux integral is if $\vec{F}$ represents some sort of fluid flow, and we place a net into our fluid in the shape of some surface S. Then, the flux of $\vec{F}$ across S measures how much our fluid is travelling through the net. If the surface is perpendicular to the direction of the fluid flow, more of the fluid is travelling through the surface, so our flux is higher, whereas if the surface is more parallel to the direction of the fluid flow, the flux will be lower. This can be measured by examining the dot product of the normal vector to the surface with the vector representing the flow of the fluid.

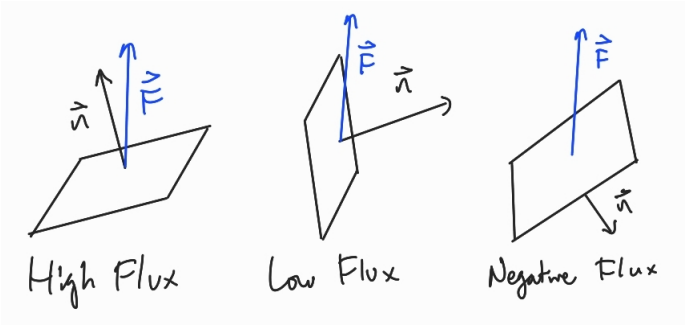


Figure 62: Configurations where flux is high, low, or even negative.

If we want to find the total amount of flux through the surface, we use our usual calculus trick: find an expression for the flux through a small chunk of surface, and integrate over the entire surface. For some small chunk of the surface, the flux is given by $\vec{F} \cdot d\vec{S}$, so to find the total flux through the entire surface, we integrate

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S}$$

Note: Physics students may recognize that flux integrals play an integral role in Maxwell's equations. All four equations in their integral form contain some form of either electric or magnetic flux integral.

There is a slight issue that does need to be resolved: given a surface and a parameterization by u and v, there are two different surface normal vectors that can be chosen (if $\vec{n}$ is one such vector, the other is $-\vec{n}$). Making a choice of surface normal vector is making a choice of **orientation** on the surface, and if we can do this in a consistent way, the surface is **orientable**. Most of the surfaces that we will encounter in this course are orientable, but certain examples, such as the Möbius strip, are not orientable.

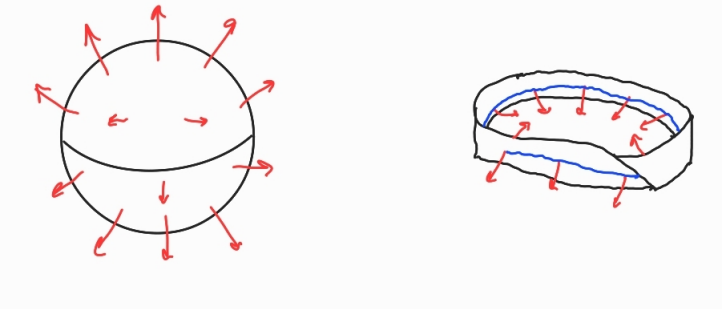


Figure 63: The sphere (left), which is orientable, versus the Möbius strip, which is not orientable.

Notice that once we choose a normal vector on the sphere, no matter how we move it around, it will never point inward. However, on the Möbius strip, we can move the normal vector along the blue path to obtain the opposite orientation at the same point, so the Möbius strip is not orientable. Generally (if not otherwise specified), when our surface is orientable, we will choose the outward facing normal vector.

Example. Let S be the region bounded by the paraboloid $1 - x^2 - y^2$ and $z = 0$. Let

$$\vec{F} = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Determine the flux of $\vec{F}$ through S, with the orientation given by the outward pointing normal vector.

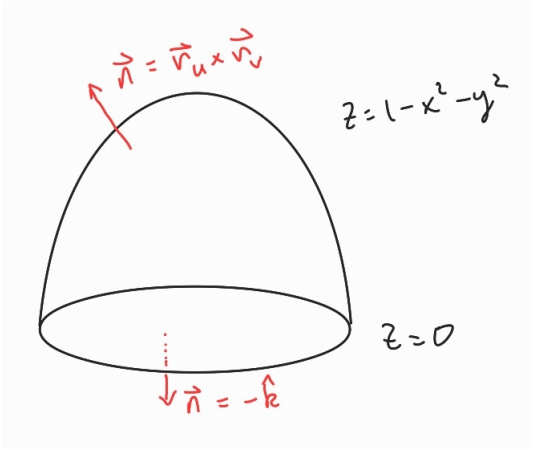


Figure 64: The given region, together with the surface normal vectors.

We first note that the flux through S is given by the flux through two surfaces: the paraboloid, which we will denote S_1, and the flat base, which we will denote S_2. Since S_1 is the graph of the function $f(x, y) = 1 - x^2 - y^2$, we can parameterize using x and y, and so we have that

$$dS = -\frac{\partial f}{\partial x}\hat{\mathbf{i}} - \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}\, dA = 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + \hat{\mathbf{k}}\, dA$$

Note that the base is the unit disc $\{(x, y) : x^2 + y^2 \leq 1\}$. We then have that

$$\begin{aligned}
\iint_{S_1} \vec{F} \cdot d\vec{S} &= \iint_S (y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + \hat{\mathbf{k}})\, dA \\
&= \iint_D 2xy + 2xy + z\, dA \\
&= \iint_D 4xy + 1 - x^2 - y^2\, dA \\
&= \int_0^{2\pi} \int_0^1 (4r^2 \sin(\theta)\cos(\theta) + 1 - r^2) r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 4r^3 \sin(\theta)\cos(\theta) + r - r^3\, dr\, d\theta \\
&= \int_0^{2\pi} \sin(\theta)\cos(\theta) + \frac{1}{2} - \frac{1}{4}\, d\theta \\
&= \frac{1}{4}(2\pi) \\
&= \frac{1}{2}\pi
\end{aligned}$$

Along the base, we have that x and y also parameterize the graph of the constant function 0, and so we have that

$$dS = -\frac{\partial f}{\partial x}\hat{\mathbf{i}} - \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}\, dA = \hat{\mathbf{k}}\, dA$$

However, this provides us the inward-facing normal vector, so we instead choose $\vec{n} = -\hat{\mathbf{k}}$. We then have that

$$\begin{aligned}\iint_{S_2} \vec{F} \cdot d\vec{S} &= \iint_S (y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (-\hat{\mathbf{k}})\, dA \\ &= \iint_D -z\, dA \\ &= \iint_D 0\, dA \\ &= 0\end{aligned}$$

We thus conclude that

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \frac{\pi}{2} + 0 = \boxed{\frac{\pi}{2}}$$

Note: From the picture, it should be fairly clear that the normal direction for S_2 is $-\hat{\mathbf{k}}$, but the magnitude may not be as clear. For example, why should we not choose $\vec{n} = -2\hat{\mathbf{k}}$? The answer is that the magnitude of $\vec{n}$ tells us how much larger the surface area above a small chunk of area is, relative to the small chunk of area. Since the surface area dS refers to precisely the same area as the dA it lies above, the scaling factor to choose should be 1, and thus $\vec{n}$ must have magnitude 1, so we choose $\vec{n} = -\hat{\mathbf{k}}$.

Example. Let S be the surface parameterized by

$$x(u,v) = 2uv \qquad y(t) = u^2 - v^2 \qquad z(t) = u^2 + v^2$$

on the domain $D = \{(u,v) \,:\, u^2 + v^2 \leq 1\}$. Let

$$\vec{F} = 2z\hat{\mathbf{i}} + x\hat{\mathbf{k}}$$

Compute

$$\iint_S \vec{F} \cdot d\vec{S}$$

We first compute

$$\begin{aligned}\vec{r}_u &= 2v\hat{\mathbf{i}} + 2u\hat{\mathbf{j}} + 2u\hat{\mathbf{k}} \\ \vec{r}_v &= 2u\hat{\mathbf{i}} - 2v\hat{\mathbf{j}} + 2v\hat{\mathbf{k}} \\ \vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2v & 2u & 2u \\ 2u & -2v & 2v \end{vmatrix} \\ &= 8uv\hat{\mathbf{i}} + 4(u^2 - v^2)\hat{\mathbf{j}} - 4(u^2 + v^2)\hat{\mathbf{k}}\end{aligned}$$

We then integrate

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_D 16(u^2 + v^2)uv - 8(u^2 + v^2)uv\, dA \\ &= \iint_D 8uv(u^2 + v^2)\, dA \\ &= 8\int_0^1 \int_0^{2\pi} r^2(r^2\cos(\theta)\sin(\theta))r\, d\theta\, dr \\ &= 8\left(\int_0^1 r^5\, dr\right)\left(\int_0^{2\pi} \cos(\theta)\sin(\theta)\, d\theta\right) \\ &= \boxed{0}\end{aligned}$$

since the θ integral will be 0.

19 Stokes' Theorem

Textbook Reference: Chapter 16.8.
Planned Lecture Date(s): May 17, 2023.

When we introduced the idea of the curl of a vector field, we restated Green's Theorem in the following form:

$$\oint_{\partial D} \vec{F} \cdot \vec{dr} = \iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dA$$

where we think of D as some region in the xy-plane. However, regions in the xy-plane are very simple cases of surfaces, and furthermore these surfaces have surface normal given by $\vec{n} = \hat{\mathbf{k}}$, where the upward-pointing orientation is chosen. We can again rephrase Green's Theorem as

$$\oint_{\partial S} \vec{F} \cdot \vec{dr} = \iint_S (\nabla \times \vec{F}) \cdot \vec{dS}$$

where S is taken to be the surface defined by D in the xy-plane. It turns out that this actually holds true for any surface S! This is the statement of **Stokes' Theorem**. This is yet another generalization of the fundamental theorem of calculus: Stokes' Theorem tells us that for some vector field $\vec{F}$, integrating the curl over the entire surface is equivalent to integrating $\vec{F}$ around the boundary. When we are integrating this special class of vector fields, we can relate the integral along the boundary to the integral throughout the surface.

When using Stokes' Theorem, we choose the direction on ∂S that satisfies the right-hand rule. For example, if the normal vector on S is facing upwards, then we choose the counter-clockwise orientation on ∂S. This was seen in Green's Theorem, where $\vec{k}$ is the upwards-pointing normal vector, and the integral over the boundary is taken counter-clockwise.

Example. Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ which lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane, and let

$$\vec{F} = xz\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}$$

Compute

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{dS}$$

using the upward facing normal.

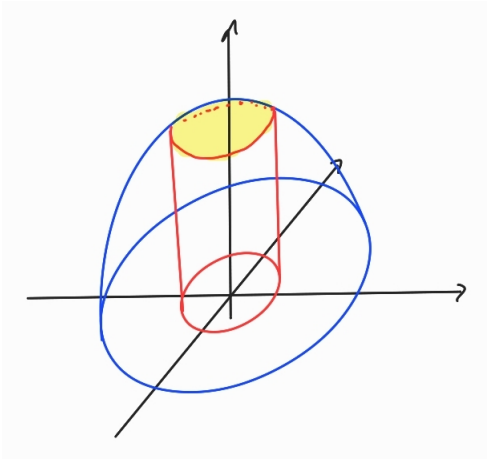


Figure 65: The given region, together with the surface normal vectors.

Since the integrand is the curl of a vector field, we proceed with Stokes' Theorem. Note that ∂S is given by

$$\begin{aligned} x^2 + y^2 + z^2 &= 4 \\ x^2 + y^2 &= 1 \\ \implies z^2 &= 3 \end{aligned}$$

and thus our boundary curve is the circle of radius 1, at $z = \sqrt{3}$. We can parameterize this as

$$x(t) = \cos(t) \qquad y(t) = \sin(t) \qquad z(t) = \sqrt{3}$$

We then have that

$$\begin{aligned} \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} &= \oint_{\partial S} \vec{F} \cdot d\vec{r} \\ &= \oint_{\partial S} xz\,dx + yz\,dy + xy\,dz \\ &= \int_0^{2\pi} \sqrt{3}\cos(t)(-\sin(t))\,dt + \sqrt{3}\sin(t)\cos(t)\,dt \\ &= \sqrt{3}\int_0^{2\pi} 0\,dt \\ &= \boxed{0} \end{aligned}$$

Immediately, we have several applications of Stokes' Theorem. For example, if S_1 and S_2 are two surfaces which share a boundary, then we have that

$$\iint_{S_1} (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S_1} \vec{F} \cdot d\vec{r} = \oint_{\partial S_2} \vec{F} \cdot d\vec{r} = \iint_{S_1} (\nabla \times \vec{F}) \cdot d\vec{S}$$

and so we see that if we are integrating a vector field which is the curl of another vector field, the surface integrals are in fact *independent of surface*! This should be thought of in analogy with the idea that path integrals of conservative vector fields are independent of path.

Another consequence of this is when S be a closed surface - that is, S has no boundary. Examples of closed surfaces include the sphere, the torus, or a cubical region. Then, if $\vec{F}$ is a vector field, we have that

$$\oiint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r} = 0$$

since we are integrating over an empty set. Note that the symbol $\oiint$ is used to note the integral over a closed surface. This should be thought of in analogy with the path integrals of conservative vector fields over loops being 0.

We can use the property that the integrals are independent of surface to simplify lots of flux integrals, especially if they are taken over surfaces that are difficult to parameterize. You can visualize this as if you are blowing bubbles with a circular wire: after you dip the wire into the bucket of soap, the soap forms the flat surface with the wire as its boundary. As you blow on the surface, the bubble grows larger and larger, but we crucially observe that any flux through the original bubble surface would still have to pass through the new, larger bubble surface.

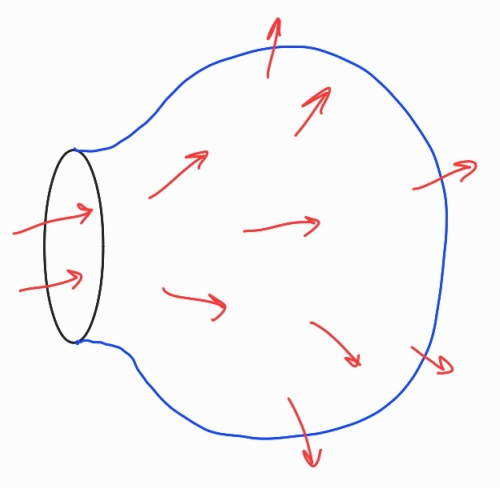


Figure 66: Any flux through the black surface also must go through the blue surface.

Example. Let S be the surface of the paraboloid $9 - x^2 - y^2$ above the xy-plane, and let

$$\vec{F} = (2x^2 + e^{y^2 z})\hat{\mathbf{i}} + (\sin(xz^3) - 4xy)\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Compute

$$\iint_S \vec{F} \cdot d\vec{S}$$

using the upwards-facing normal.

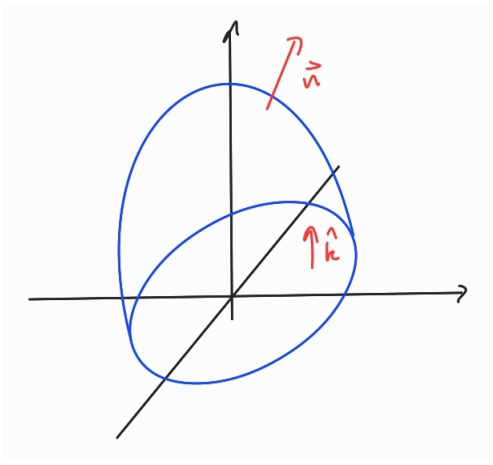


Figure 67: The surface S, bounded by the circle of radius 3 in the xy-plane.

We first compute

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x^2 + e^{y^2 z}) + \frac{\partial}{\partial y}(\sin(xz^3) - 4xy) + \frac{\partial}{\partial z}1 = 4x - 4x = 0$$

and since the vector field is defined on all of $\mathbb{R}^3$, which does not have holes, we know that $\vec{F}$ must be the curl of some other vector field. Thus, surface integrals of $\vec{F}$ must be independent of surface, so instead of integrating over S, we'll integrate over the disk of radius 3 in the xy-plane, which we'll denote D. Note

that the surface normal to D is simply $\hat{\mathbf{k}}$. We then have

$$\begin{aligned}\iint_S \vec{F}\cdot d\vec{S} &= \iint_D \vec{F}\cdot d\vec{S}\\ &= \iint_D ((2x^2+e^{y^2z})\hat{\mathbf{i}}+(\sin(xz^3)-4xy)\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot\hat{\mathbf{k}}\,dA\\ &= \iint_D 1\,dA\\ &= \boxed{9\pi}\end{aligned}$$

This saved us a tremendous amount of work, since we were able to integrate over a much simpler region.

If $\vec{F}=\nabla\times\vec{A}$ for some vector field $\vec{A}$, we say that $\vec{A}$ is a **vector potential** for $\vec{F}$. In general, finding a vector potential is a fairly difficult task (and generally not unique up to a constant), but if we are able to find a vector potential, we can also use Stokes' Theorem to save us a lot of work.

Example. Let

$$\vec{F}=2z\hat{\mathbf{i}}+2x\hat{\mathbf{j}}$$

Show that $\vec{F}$ can be written as $\nabla\times\vec{A}$, and find $\vec{A}$.

We check that

$$\nabla\cdot\vec{F}=0+0=0$$

and since $\vec{F}$ is defined on all of $\mathbb{R}^3$, which has no holes, some vector potential $\vec{A}$ must exist. We see that if

$$\vec{A}=P\hat{\mathbf{i}}+Q\hat{\mathbf{j}}+R\hat{\mathbf{k}}$$

then we have that

$$\vec{F}=\nabla\times\vec{A}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\hat{\mathbf{j}}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}$$

We then extract the relations

$$\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=2z \qquad \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}=2x \qquad \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0$$

and at this stage, we need to use some trial and error. If we guess that $\frac{\partial R}{\partial y}=0$, from the first equation, we see that $Q=-z^2+C_1(x,y)$. If we further assume that $C_1(x,y)=0$, then using the third equation, we see that $\frac{\partial P}{\partial y}=0$, so $P=C_2(x,z)$. Further assuming $C_2(x,z)=0$, we see from the second equation that $\frac{\partial R}{\partial x}=-2x$, so choosing $R=-x^2+C_3(y,z)$ and taking $C_3(y,z)=0$ yields a solution given by

$$\boxed{\vec{A}=z^2\hat{\mathbf{j}}+x^2\hat{\mathbf{k}}}$$

This solution is very much not unique, and we were very fortunate that the assumptions we made along the way were indeed valid, and did not obstruct us from finding a solution. Finding a vector potential, in general, is a very difficult process.

Note: Stokes' Theorem is named after George Stokes, and so the apostrophe in the name does indeed go at the very end. It turns out that the fundamental theorem of calculus, the fundamental theorem of line integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem from next lecture are all specific examples of a more general statement relating objects called **differential forms**. The discussion of this is beyond the scope of this course, but be aware that the name "Stokes' Theorem" is often used to refer to this more general statement, and so if you are reading about these topics, any of the "fundamental theorems" that we have covered may possibly be referred to as "Stokes' Theorem".

20 Divergence Theorem

Textbook Reference: Chapter 16.9.
Planned Lecture Date(s): May 19, 2023.

Just as we can relate two dimensional integrals to their one dimensional boundaries, we can do the same with three dimensional integrals and their two dimensional boundaries, provided that we set things up the right way. Let's think about the tools we have.

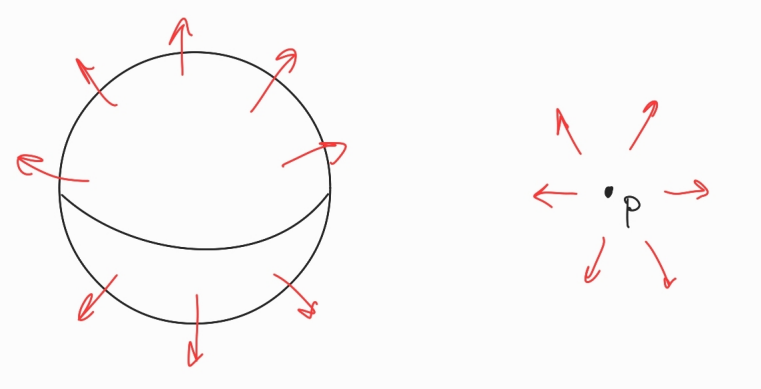


Figure 68: The red vector field going through the sphere (left), and the divergence of the red vector field at a point p (right).

- On one hand, given a surface, we can compute the flux of a vector field $\vec{F}$ through a surface S. If we require that the surface does not have a boundary, which we will call a **closed** surface, then the flux of $\vec{F}$ through S represents how much the vector field is "leaving" the region that is bounded by S.
- On the other hand, we have a way to measure how much a vector field $\vec{F}$ is expanding at any point, given by the divergence of $\vec{F}$, denoted $\nabla \cdot \vec{F}$.

These two ideas should feel related, as we think of points where the divergence is nonzero as a source (when $\nabla \cdot \vec{F}$ is positive, otherwise it would be a sink) of the vector field. If my closed surface encloses a source, then the overall "flow" of the vector field should leave the region bounded by the surface. Similarly, if my closed surface encloses a sink, then the overall "flow" of the vector field should enter the region bounded by the surface. Finally, if the surface does not enclose any points of nonzero divergence, then the net flow into the surface should be equal to the net flow out, as the flow has nowhere to go, and thus the total flux should be 0.

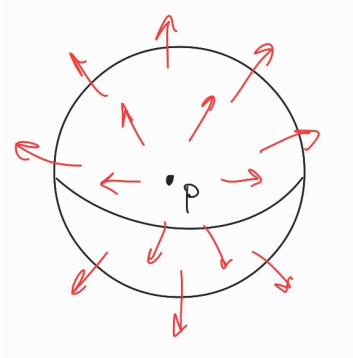


Figure 69: If p is inside the region bounded by the surface, then the flux through the surface should be related to the divergence at p.

This is the idea behind the **Divergence Theorem**. If we integrate the flux over a closed surface, this should be the same as summing up the total divergence inside. Mathematically, we can phrase this in the following way. Let E be a region of space, and let ∂E be its boundary. Then, we have that

$$\oiint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \nabla \cdot \vec{F}\, dV$$

where the outward orientation is chosen on ∂E. Note that just as we needed simple curves for Green's Theorem and Stokes' Theorem to hold, we need our surface ∂E to be orientable to make sense of an interior and exterior. The symbol $\oiint$ means that the surface integral we are taking is over a closed surface.

We can make a few immediate conclusions from the Divergence Theorem. The first is that if $\nabla \cdot \vec{F} = 0$, or in other words, $\vec{F}$ is divergence free, then every surface integral of $\vec{F}$ will be 0. Furthermore, on a domain without holes, $\nabla \cdot \vec{F} = 0$ precisely when $\vec{F} = \nabla \times \vec{A}$ for some vector potential $\vec{A}$. Thus, we have that

$$\oiint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iiint_E \nabla \cdot (\nabla \times \vec{F})\, dV = \iiint_E 0\, dV = 0$$

for any closed surface S. This can also be seen from Stokes' Theorem, as we have that

$$\oiint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

where the second integral is 0 since ∂S is empty, as S is a closed surface.

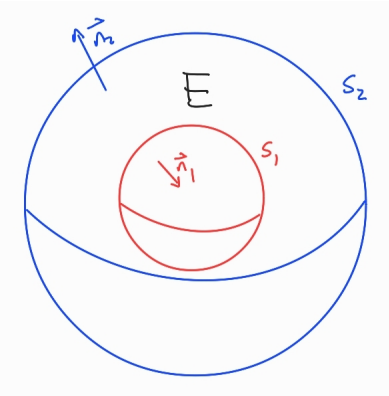


Figure 70: Two surfaces S_1 and S_2, where S_1 completely sits inside the region bounded by S_2.

We can come up with a similar analogue of independence of path for surface integrals, in the following way. Let E be the region between S_1 and S_2. Then, the boundary of E is given by S_1, with the usual orientation, but S_2 with the opposite orientation, since outwards from E is inwards if we think of S_2 separately. Thus, for some vector field $\vec{F}$, we have that

$$\begin{aligned}\oiint_{S_1} \vec{F} \cdot d\vec{S} - \oiint_{S_1} \vec{F} \cdot d\vec{S} &= \oiint_{\partial E} \vec{F} \cdot d\vec{S} \\ &= \iiint_E (\nabla \cdot \vec{F})\, dV\end{aligned}$$

and so if $\vec{F}$ does not diverge in E, the region between S_1 and S_2, then the flux integrals over S_1 and S_2 will differ by 0, and therefore be the same. The intuition we should have (analogous to Green's Theorem with the curl) is that throughout regions where $\nabla \cdot \vec{F} = 0$, we can deform our surfaces at will without changing the value of the flux integral.

Example. Find the flux of the vector field

$$\vec{F} = z\hat{\mathbf{i}} + (x^2 + 2x)y\hat{\mathbf{j}} + x\hat{\mathbf{k}}$$

through the unit sphere $x^2 + y^2 + z^2 = 1$.

We compute that

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} z + \frac{\partial}{\partial y}(x^2 + 2x)y + \frac{\partial}{\partial z} x = 0 + x^2 + 2x + 0 = x^2$$

Let B denote the unit ball, and let $S = \partial B$ be the unit sphere. We then have that

$$\begin{aligned}
\oiint_S \vec{F} \cdot d\vec{S} &= \iiint_B \nabla \cdot \vec{F}\, dV \\
&= \iiint_B x^2 + 2x\, dV \\
&= \iiint_B x^2\, dV \qquad \text{(odd function over symmetric region)} \\
&= \int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \cos(\theta)^2 \sin^2(\varphi) \rho^2 \sin(\varphi)\, d\varphi\, d\theta\, d\rho \\
&= \left(\int_0^1 \rho^4\, d\rho \right) \left(\int_0^{2\pi} \cos^2(\theta)\, d\theta \right) \left(\int_0^{\pi} \sin^3(\varphi)\, d\varphi \right) \\
&= \frac{1}{5}\pi \left(\int_{-1}^1 1 - u^2\, du \right) \\
&= \frac{1}{5}\pi \left(\frac{4}{3} \right) \\
&= \boxed{\frac{4}{15}\pi}
\end{aligned}$$

However, we haven't really utilized the full power of the Divergence Theorem yet. As we might recall, surfaces can often be hard to parameterize, but the regions they contain may be significantly easier.

Example. Find the flux of the vector field

$$\vec{F} = 2x\hat{\mathbf{i}} + 3xy\hat{\mathbf{j}} + xz\hat{\mathbf{k}}$$

through the unit cube, given by

$$S = \{(x, y, z) \,:\, 0 \leq x, y, z \leq 1\}$$

We compute

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} 2x + \frac{\partial}{\partial y} 3xy + \frac{\partial}{\partial z} xz = 2 + 3x + x = 4x + 2$$

Let E be the interior of the cube. We then have that

$$\begin{aligned}
\oiint_S \vec{F} \cdot d\vec{S} &= \iiint_E \nabla \cdot \vec{F}\, dV \\
&= \iiint_E 4x + 2\, dV \\
&= \int_0^1 \int_0^1 \int_0^1 4x + 2\, dz\, dy\, dx \\
&= \int_0^1 4x + 2\, dx \\
&= \boxed{4}
\end{aligned}$$

Note that if we had tried to compute the flux integrals directly, we would have had to parameterize six surfaces, but instead we were able to get away with one simple computation.

We can also revisit an earlier computation, from when we first encountered surface integrals.

Example. Let S be the surface of the paraboloid $9 - x^2 - y^2$ above the xy-plane, and let

$$\vec{F} = (2x^2 + e^{y^2 z})\hat{\mathbf{i}} + (\sin(xz^3) - 4xy)\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Compute

$$\iint_S \vec{F} \cdot d\vec{S}$$

using the upwards-facing normal.

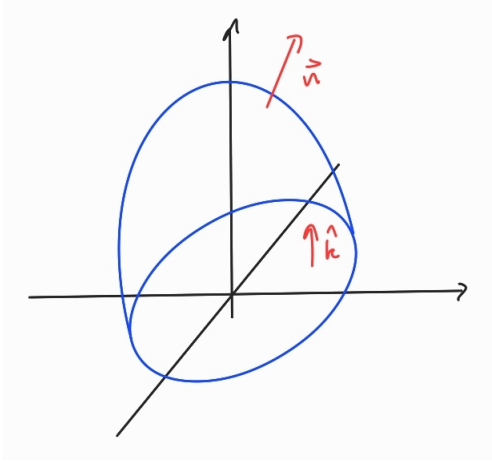


Figure 71: The surface S, bounded by the circle of radius 3 in the xy-plane.

Just as before, we first compute

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x^2 + e^{y^2 z}) + \frac{\partial}{\partial y}(\sin(xz^3) - 4xy) + \frac{\partial}{\partial z}1 = 4x - 4x = 0$$

Then, we instead make the following argument. We have that S is the parabolic portion of the surface, and we let S_1 denote the flat base of the surface, both taken with the upward pointing normal vector. The surfaces S and S_1 together form the closed surface S', which encloses a region, call it E. Then, we have that

$$\begin{aligned} 0 &= \iiint_E \nabla \cdot \vec{F}\, dV \\ &= \oiint_{S'} \vec{F} \cdot d\vec{S} \\ &= \iint_S \vec{F} \cdot d\vec{S} - \iint_{S_1} \vec{F} \cdot d\vec{S} \end{aligned}$$

where the integral over S_1 carries a minus sign, as the outward pointing normal from E is the downwards facing normal on S_1. Thus, we can similarly conclude that the integrals over the parabolic surface and the base surface are the same. Note however that this is slightly more powerful than before, as even if $\nabla \cdot \vec{F} \neq 0$, we could still relate the surface integral over the parabolic region to the surface integral over the base using the relationship

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iiint_E \nabla \cdot \vec{F}\, dV$$

where the volume integral of the divergence can be thought of as a "correction term" when changing surfaces of integration.

21 Recap and Examples

Textbook Reference: Chapter 16.10.
Planned Lecture Date(s): May 20, 2023.

Let's do a quick summary of the tools we've developed in vector calculus.

- **Fundamental Theorem of Calculus:** Let $F(x)$ be a differentiable function. Then, we have that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

- **Fundamental Theorem of Line Integrals:** Let f be a function, and let C be a path parameterized by $\vec{r}(t)$ with $t \in [a, b]$. Then, we have that

$$\int_C \nabla f \cdot \vec{dr} = f(\vec{r}(b)) - f(\vec{r}(a))$$

- **Green's Theorem:** Let D be a region in the plane, with boundary curve $C = \partial D$, with C parameterized counter-clockwise by $\vec{r}(t)$, and let $\vec{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$. Then, we have that

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \oint_C \vec{F} \cdot \vec{dr}$$

- **Stokes' Theorem:** Let S be a surface with boundary curve $C = \partial S$, parameterized counterclockwise by $\vec{r}(t)$, and let $\vec{F}$ be a vector field. Then, we have that

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{dS} = \oint_C \vec{F} \cdot \vec{dr}$$

- **Divergence Theorem:** Let E be a region with boundary surface $S = \partial E$, taken with the outward orientation, and let $\vec{F}$ be a vector field. Then, we have that

$$\iiint_E (\nabla \cdot \vec{F})\,dV = \oiint_S \vec{F} \cdot \vec{dS}$$

From this viewpoint, we can see that these are all analogous formulations of the same concept: in order to compute the integral of some quantity over an n-dimensional region, we can compute the integral of a related "derivative" quantity over the $(n-1)$-dimensional boundary.

$$\begin{array}{ccc}
\{\text{Functions}\} & \longrightarrow & \{0\text{-D}\} \\
{\scriptstyle f \mapsto \nabla f}\downarrow & & \uparrow{\scriptstyle \partial} \\
\{\text{Vector Fields}\} & \longrightarrow & \{1\text{-D}\} \\
{\scriptstyle \vec{F} \mapsto \nabla \times \vec{F}}\downarrow & & \uparrow{\scriptstyle \partial} \\
\{\text{Vector Fields}\} & \longrightarrow & \{2\text{-D}\} \\
{\scriptstyle \vec{F} \mapsto \nabla \cdot \vec{F}}\downarrow & & \uparrow{\scriptstyle \partial} \\
\{\text{Functions}\} & \longrightarrow & \{3\text{-D}\}
\end{array}$$

If we go back to our diagram relating ∇ operations to each other, we see that along the left side, going down along any two arrows gives 0. Similarly, along the right side, taking the boundary twice gives 0. Furthermore, we can relate integrals at one level to integrals at the adjacent levels. However, there is a subtlety: in order to move from one level to an adjacent level, we have to traverse one arrow in the forward

direction, which we can always do, but we have to traverse one arrow in the reverse direction, which we cannot always do. If we find a way to traverse a "backwards" arrow on the left, which amounts to finding an anti-derivative, potential function, or vector potential, then we can relate the integral to an integral in one dimension lower via one of our fundamental theorems. Similarly, if the region we are integrating over can be thought of as the boundary of some higher dimensional region, we can traverse an arrow "backwards" on the right to relate our integral to an integral over one higher dimension. This consolidates all of the theorems we have seen into one unified picture.

To illustrate the uses of all of the tools that we have developed, we'll compute a few examples.

Example. Find the work done by the vector field

$$\vec{F} = (4x^3y^2 - 2xy^3)\hat{\mathbf{i}} + (2x^4y - 3x^2y^2 + 4y^3)\hat{\mathbf{j}}$$

along the path

$$\vec{r}(t) = \begin{pmatrix} t + \sin(\pi t) & 2t + \cos(\pi t) \end{pmatrix} \qquad t \in [0, 1]$$

We first check to see if $\vec{F}$ is conservative. Since it is defined on the entire plane, which is simply connected, it is enough to check that the partial derivative condition is satisfied. We compute

$$\begin{aligned} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} &= \frac{\partial}{\partial x}(2x^4y - 3x^2y^2 + 4y^3) - \frac{\partial}{\partial y}(4x^3y^2 - 2xy^3) \\ &= 8x^3y - 6xy^2 - 8x^3y + 6xy^2 \\ &= 0 \end{aligned}$$

This is great news, since this means that a potential function exists. We compute

$$\begin{aligned} \int (4x^3y^2 - 2xy^3)\, dx &= x^4y^2 - x^2y^3 + C_1(y) \\ \int (2x^4y - 3x^2y^2 + 4y^3)\, dy &= x^4y^2 - x^2y^3 + y^4 + C_2(x) \end{aligned}$$

We compare terms and see that a potential function is given by

$$f(x, y) = x^4y^2 - x^2y^3 + y^4$$

We then see that

$$\vec{r}(0) = (0, 0) \qquad \vec{r}(1) = (1, 1)$$

and so by the fundamental theorem of line integrals, we have that

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= f(1, 1) - f(0, 0) \\ &= (1 - 1 + 1) - (0 - 0 + 0) \\ &= \boxed{1} \end{aligned}$$

Example. Let $\vec{F} = x^2y\hat{\mathbf{i}} + x^2y^3\hat{\mathbf{j}}$. Evaluate

$$\oint_C \vec{F} \cdot d\vec{r}$$

where C is the circle of radius 1, oriented clockwise.

Since the contour C is a closed loop, we can apply Green's (or Stokes') Theorem. We compute

$$\nabla \times \vec{F} = (2xy^3 - x^2)\hat{\mathbf{k}}$$

and so we instead integrate over the interior of C (call it D). We have

$$\begin{aligned}\oint_C \vec{F}\cdot\vec{dr} &= -\iint_D (\nabla\times\vec{F})\cdot\hat{\mathbf{k}}\,dA\\ &= -\iint_D 2xy^3 - x^2\,dA\end{aligned}$$

Since $2xy^3$ is odd in x (and y) and D is symmetric in x (and y), that term will integrate to 0, so we are left with

$$\begin{aligned}&= \iint_D x^2\,dA\\ &= \int_0^1\int_0^{2\pi} r^2\cos^2(\theta) r\,d\theta\,dr\\ &= \boxed{\frac{1}{4}\pi}\end{aligned}$$

where the minus sign comes from the clockwise orientation.

Example. Compute the surface integral

$$\iint_S (\nabla\times\vec{F})\cdot\vec{dS}$$

where

$$\vec{F} = xy\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + zx\hat{\mathbf{k}}$$

and S is the part of the sphere $x^2+y^2+z^2=5$ that lies above the plane $z=1$, and S is oriented upwards.

We see that since we are integrating the curl of $\vec{F}$ over a surface, we can apply Stokes' Theorem to instead integrate $\vec{F}$ over the boundary. We see that the boundary is given by when $z=1$ intersects $x^2+y^2+z^2=5$, and so the intersection is a circle of radius 2 in the plane $z=1$, centered at the origin. We thus parameterize ∂S by

$$x(t) = 2\cos(t) \qquad y(t) = 2\sin(t) \qquad z = 1$$

noticing that this parameterization will go around ∂S in a counterclockwise direction. We then have that

$$\begin{aligned}\iint_S \nabla\times\vec{F}\cdot\vec{dS} &= \oint_{\partial S}\vec{F}\cdot\vec{dr}\\ &= \oint_{\partial S} xy\,dx + yz\,dy + zx\,dz\\ &= \int_0^{2\pi} 4\cos(t)\sin(t)(-2\sin(t))\,dt + 2\sin(t)(2\cos(t))\,dt\\ &= \int_0^{2\pi} -8\cos(t)\sin^2(t) + 4\sin(t)\cos(t)\,dt\\ &= \boxed{0}\end{aligned}$$

where you can u-substitute for the last integral with $u=\sin(t)$, but find that $\sin(2\pi)=\sin(0)$ so the integral with respect to u will be over an interval of length 0.

Example. Compute the flux of the vector field

$$\vec{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$$

through the surface given by the boundary of the cylinder $E=\{(x,y,z)\,:\,x^2+y^2\le 1, 0\le z\le 2\}$, with the outward facing orientation.

By the Divergence Theorem, it is enough to integrate the divergence of $\vec{F}$ over E. We compute

$$\nabla \cdot \vec{F} = 1 + 1 + 2z = 2z + 2$$

and so we have that

$$\begin{aligned}
\oiint_{\partial E} \vec{F} \cdot d\vec{S} &= \iiint_E \nabla \cdot \vec{F}\, dV \\
&= \iiint_E 2z + 2\, dV \\
&= \int_0^1 \int_0^{2\pi} \int_0^2 (2z+2) r\, dz\, d\theta\, dr \\
&= 2\pi \left(\int_0^1 r\, dr \right) \left(\int_0^2 2z + 2\, dz \right) \\
&= 2\pi \left(\frac{1}{2} \right) (8) \\
&= \boxed{8\pi}
\end{aligned}$$

A Complex Variables

Planned Lecture Date(s): May 22, 2023.

One of the mathematical applications of multivariable comes in the calculus of complex variables. So far, all of our calculus is done over $\mathbb{R}$, the real numbers. However, in some sense, the real numbers are not all we care about: if you've taken linear algebra or differential equations, complex numbers sometimes show up in finding eigenvalues or solving second order linear differential equations. The cause of this is because they arise as roots of polynomials! If we write down a polynomial with real coefficients, their roots might not be real. For example, the polynomial

$$f(x) = x^2 + 1$$

has no real roots, since any root r would satisfy $r^2 = -1$, but the square of any real number is positive. As such, we give ourselves a symbol i, which we essentially declare to be a solution to the equation $x^2 + 1$. At this point, the following miracle occurs: by simply adding this one "symbol" to the real numbers (and allowing us to add and multiply this new symbol with the appropriate properties), we can actually solve *any* polynomial with real coefficients!

The set of numbers that we have created is denoted $\mathbb{C}$, the complex numbers. Any sum and product of our symbol i with numbers in $\mathbb{R}$ can be grouped by powers of i, and furthermore, since $i^2 = -1$, which is an element of $\mathbb{R}$, any powers of i higher than 2 can be rewritten (by factoring out i^2 repeatedly) as a real number times either $i^0 = 1$ or $i^1 = i$. Thus, we have the following representation of the complex numbers.

$$\mathbb{C} = \{x + iy \,:\, x, y \in \mathbb{R}\}$$

We can add two complex numbers using the rule

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

If you have taken a course in linear algebra, you may notice that this makes $\mathbb{C}$ look like $\mathbb{R}^2$, which is a vector space over $\mathbb{R}$. However, notice that we have more structure than a vector space, since we can additionally multiply complex numbers (which we cannot do with vectors!).

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

We'll usually use the variable z to denote a complex variable, so we will frequently write z, which will be assumed to mean $z = x + iy$. We will also denote $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$, and draw $\mathbb{C}$ as a two dimensional plane, with the real part on the x-axis and the imaginary part on the y-axis.

Of course, defining the complex numbers on their own is not of much use, so we should also define functions of complex variables. The second miracle that occurs is the following. For a *real* variable t, we have that

$$\begin{aligned}
e^{it} &= 1 + (it) + \frac{1}{2!}(it)^2 + \frac{1}{3!}(it)^3 + \cdots + \frac{1}{n!}(it)^n + \cdots \\
&= \left(1 + \frac{1}{2!}(it)^2 + \frac{1}{4!}(it)^4 + \cdots\right) + \left((it) + \frac{1}{3!}(it)^3 + \frac{1}{5!}(it)^5 + \cdots\right) \\
&= \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \cdots\right) + \left(it - i\frac{1}{3!}t^3 + i\frac{1}{5!}t^5 - \cdots\right) \\
&= \cos(t) + i\sin(t)
\end{aligned}$$

where we use the Taylor series of e^x, $\cos(x)$, and $\sin(x)$. This gives us an interesting way to encapsulate complex numbers: since $\cos(t) + i\sin(t)$ defines a circle (where t is the angle), we can use **polar form** to write

$$x + iy = Re^{it}$$

where (R, t) are analogous to (r, θ) of polar coordinates. Just as in polar coordinates, every non-zero point has a unique representation, up to a multiple of 2π in t.

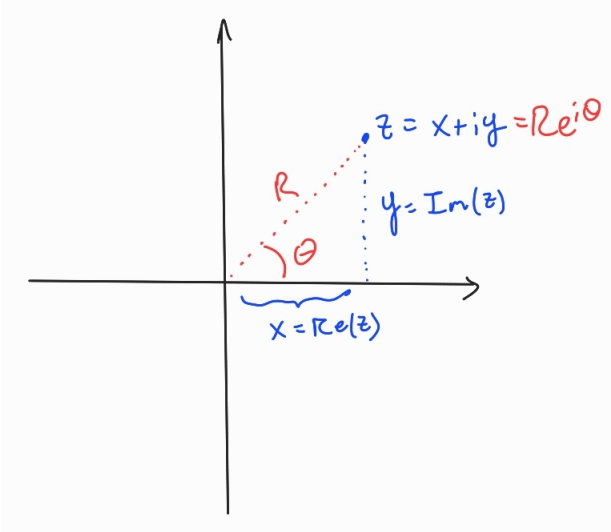


Figure 72: Cartesian and polar representations of a complex number.

We can also define an absolute value (or **norm**) of complex numbers, given by

$$\begin{aligned} |z| &= |x+iy| = \sqrt{x^2+y^2} \\ &= \left|Re^{i\theta}\right| = R \end{aligned}$$

which extends our usual absolute value in $\mathbb{R}$. Note that the equation $|z - z_0| = R$ defines a circle of radius R around z_0.

Since we are interested in the calculus of complex variables, we should investigate how to take derivatives of functions of complex variables. Let $f(z)$ be a function of complex variables. Then, if we try to set up the same limit definition from Calc II, we have

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

provided that the limit is defined. If this limit does indeed exist, we say that f is **differentiable**. This is where the subtlety comes in: whereas in one real variable, we can only approach from the left or from the right (and we need these one-sided limits to agree for the limit to exist), in our case, h is a complex number, and can approach zero from infinitely many ways in the complex plane.

Note: If you've taken real analysis, this is the key point at which the calculus of complex variables diverges from the calculus of real variables. In particular, being differentiable in the complex sense is an **incredibly** restrictive condition, since we need this limit to exist from *any* direction approaching the origin, instead of just the left and the right.

Let's see what happens when we try to approach 0 from different directions. We can break f down into a multivariable function of real variables, by slightly abusing our notation and writing

$$f(z) = f(x+iy) = f(x,y) = u(x,y) + iv(x,y)$$

where u and v represent the real and imaginary parts of f. If we force h to approach along the real axis, then we have

$$f'(z) = \lim_{t\to 0} \frac{f(z+t) - f(z)}{t} = \lim_{t\to 0} \frac{f(x+t,y) - f(x,y)}{t} = \frac{\partial f}{\partial x}$$

where t is a *real* variable. If we force h to approach along the imaginary axis, then we have

$$f'(z) = \lim_{t\to 0} \frac{f(z+it) - f(z)}{it} = \frac{1}{i} \lim_{t\to 0} \frac{f(x,y+t) - f(x,y)}{t} = -i\frac{\partial f}{\partial y}$$

where t is also a *real* variable. Since these quantities both represent the limit, we can write

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -i\frac{\partial f}{\partial y} \\
\frac{\partial}{\partial x}(u+iv) &= -i\frac{\partial}{\partial y}(u+iv) \\
\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) &= -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\
\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}
\end{aligned}$$

Since these are equal as functions, their real and imaginary parts must be equal, so we obtain the equalities

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

We conclude that any complex function $f(z)$ which is differentiable must satisfy the above equations. These equations are called the **Cauchy-Riemann equations**, and it turns out that this is both necessary and sufficient for complex differentiability.

Example. Show that the function $f(z) = z^2$ is differentiable.

We see that

$$\begin{aligned}
f(z) &= z^2 \\
&= (x+iy)^2 \\
&= x^2 + 2ixy - y^2 \\
&= (x^2 - y^2) + i(2xy)
\end{aligned}$$

and so we see that $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We check that

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

and also that

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

and so we conclude that $f(z)$ is indeed differentiable.

Note: It turns out that if a function $f(z)$ is complex differentiable at a point, it is differentiable *infinitely* many times - furthermore, $f(z)$ has a Taylor series converging on some open set containing that point (such functions are called **holomorphic** or **analytic**). This miracle result is *absurdly* strong (and very, very false in the real analysis case) and is one of the major reasons why complex analysis is a beautiful subject. The proof of this follows from Cauchy's Integral Formula, which I will not have time to discuss, but is very understandable with the material we will cover.

Most of the functions that we will care about, such as polynomials, exponentials, and trigonometric functions carry over from the real setting, as we can essentially define them using their *real* Taylor series, and show that the series converges in the complex setting. For example, we have that the exponential function is given by

$$e^z = \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

and it turns out that this function is differentiable (and therefore holomorphic) on all of $\mathbb{C}$ (such functions are called **entire**). We can similarly define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{iz}}{2i}$$

and we see that all of the exponential and trigonometric identities from the real variable case still hold.

When migrating our series over from the real case, we need to worry a little bit about the convergence of Taylor series. For example, recall that the series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

centered at $x = 0$ converges whenever $|x| < 1$, and in fact this also holds when x is replaced with a complex variable z. This makes sense, since at $x = 1$, we are dividing by 0, which doesn't make sense in this context. Thus, we imagine an interval centered at 0, growing outwards until it hits a point at which the Taylor series is no longer defined, at which point it stops growing. This gives us the interval of convergence, and if we do a similar operation in the complex plane, we see the disk of convergence stop growing when $|z| = 1$, as it runs into the point $z = 1$.

However, what if we instead consider the series

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

which we know, from calculus, also converges on $|x| < 1$. This time, if we try to grow our interval of convergence, we don't run into any points which cause us to divide by 0, so it seems as if we could grow this interval infinitely. The issue lies in the fact that we are not considering complex points, as the function $\frac{1}{1+z^2}$ is dividing by zero when $z = \pm i$. Thus, when we try to play the same game in the complex plane, our disk of convergence grows until it hits these two points at $z = \pm i$, and stops growing, giving us a disk of convergence of $|z| < 1$. This is interesting! The Taylor series for $\frac{1}{1+x^2}$ is purely real, but somehow "knows" about the complex points which cause issues. We'll investigate these "problem points" in more depth.

We can extend the notion of Taylor series to a broader class of functions defined in the complex plane, by also allowing negative powers in our expansion. These series are called **Laurent series**, and we say that $f(z)$ has a Laurent series centered at $z_0 \in \mathbb{C}$ if

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where the sum on the right hand side converges on some open neighborhood around z_0, with the point z_0 excluded. When all $a_n = 0$ when $n < 0$, this agrees with our notion of Taylor series, but this allows us to talk about functions which we previously could not. However, if some $a_n \neq 0$ when $n < 0$, then $f(z)$ is not defined at $z = z_0$, since the term $(z - z_0)^n$ would be dividing by zero. When this happens, we say that $f(z)$ has a **isolated singularity** at z_0. There are two cases here:

- There is some smallest (most negative) N such that $a_N \neq 0$, and for all $n < N$, $a_n = 0$. Then, we can write

  $$f(z) = \sum_{n=N}^{\infty} a_n (z - z_0)^n$$

  and we say that $f(z)$ has a **pole** of order $-N$ at z_0. For example, the function

  $$f(z) = \frac{1}{z^3} + \frac{1}{z^2}$$

  has a pole of order 3 at $z = 0$. One can think of poles as zeroes with negative multiplicity. A pole of order 1 is sometimes called a **simple pole**.

- There are infinitely many $n < 0$ such that $a_n \neq 0$, so no smallest such n exists. In this case, we say that $f(z)$ has an **essential singularity** at z_0. For example, the function

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}$$

  has an essential singularity at 0.

Laurent series give us a standard way to think about a broad class of complex functions, and we'll later see that they (together with isolated singularities) play an important role in the integration of complex functions.

One function that we cannot directly migrate to the complex case is the inverse of e^z, or the logarithm. Let's think about what we would like the complex logarithm function (which we will denote Log) to do. For some $z = Re^{i\theta}$, we would like

$$\begin{aligned} \mathrm{Log}(z) &= \mathrm{Log}\left(Re^i\theta\right) \\ &= \mathrm{Log}(R) + \mathrm{Log}\left(e^{i\theta}\right) \end{aligned}$$

Since R is real, we can simply let this be $\ln(|R|)$, and for the other term, the logarithm should "undo" exponentiation.

$$= \ln(|R|) + i\theta$$

and so we have defined a logarithm! Unfortunately, this is not quite a function, since θ is not well-defined. However, if we restrict the range of θ (this is the same process as making a branch cut from our discussion of line integrals), then $\mathrm{Log}(z)$ will be defined for all non-zero z.

B Complex Integration

Planned Lecture Date(s): May 26, 2023.

If we'd like to do calculus, we should also be able to integrate functions of complex variables. However, since the complex plane is two-dimensional, we also have to specify the path of integration. These are simply line integrals! We can compute an integral such as

$$\int_C f(z)\,dz$$

by parameterizing C using some function $\gamma(t)$, which takes in a real parameter $t \in [a,b]$ and outputs a complex number (thus drawing a curve in the complex plane). We then have that

$$\int_C f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$

in analogy with the ds-type of line integrals that we have seen.

What happens if we try to integrate a differentiable function around a closed loop? Suppose that

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

and f is differentiable. Since $z = x+iy$, we have that $dz = dx + idy$. Then, for some closed loop C with interior D, we have that

$$\begin{aligned}
\oint_C f(z)\,dz &= \oint_C (u+iv)(dx+idy) \\
&= \oint_C u\,dx + iv\,dx + iu\,dy - v\,dy \\
&= \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy
\end{aligned}$$

This looks like two work integrals, which we are familiar with, so we can apply Green's Theorem to obtain

$$\begin{aligned}
&= \iint_D \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\,dA + \iint_D \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\,dA \\
&= 0
\end{aligned}$$

since both integrands are zero by the Cauchy-Riemann equations. This is wonderful! The result is that for a function $f(z)$ and loop C such that $f(z)$ is differentiable on the interior of the loop, we have that

$$\oint_C f(z)\,dz = 0$$

and this is known as **Cauchy's Integral Theorem**.

Example. Compute

$$\int_C z^2\,dz$$

where C is the straight line path from 0 to $1+i$.

We parameterize C by $\gamma(t) = (1+i)t$, and so $\gamma'(t) = 1+i$. We then have that

$$\begin{aligned}
\int_C z^2\,dz &= \int_0^1 ((1+i)t)^2(1+i)\,dt \\
&= (1+i)^3\int_0^1 t^2\,dt \\
&= \boxed{\frac{(1+i)^3}{3}}
\end{aligned}$$

One may notice that this agrees with if we had simply taken an antiderivative of z^2, and subtracted the value at the start from the value at the end of the path, almost as if we had simply replaced the variable z with the variable x and pretended it was a Calc II integral. This is, of course, not an accident, as we have the **fundamental theorem of complex line integrals**, which states that

$$\int_C f'(z)\,dz = f(z_1) - f(z_0)$$

where C is a path starting at z_0 and ending at z_1. Note that we can use our standard Green's Theorem trick (repackaged as Cauchy's Integral Formula) to show that the integral is indeed independent of path, and working through some details we can see that the antiderivative is indeed the correct potential function to choose.

Example. Let's compute

$$\oint_C z^n\,dz$$

where C is a counter-clockwise contour around 0.

When $n \geq 0$, we can apply Cauchy's Integral Formula directly to obtain that the value of the integral is 0, as z^n for $n \geq 0$ is differentiable. When $n < 0$, we need to be a bit more careful. The function z^n for $n < 0$ is still differentiable at all $z \neq 0$, but is not differentiable at 0.

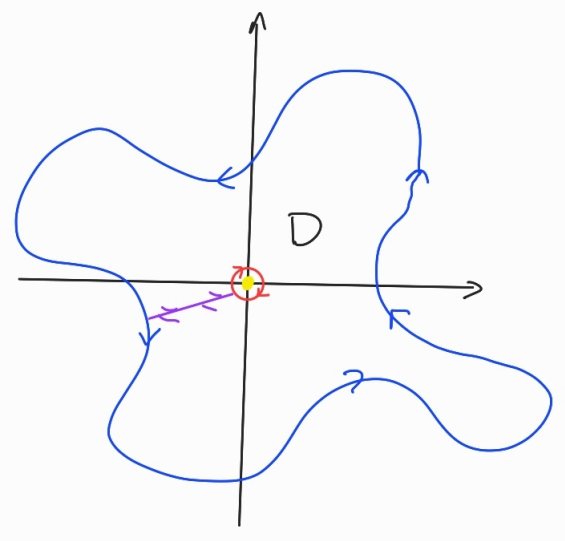


Figure 73: Integrating around the contour C is the same as integrating around the inner (red) contour.

The first thing we do is notice that we can instead integrate over a smaller contour of radius R around $z = 0$. This uses our usual path-independence trick: we traverse C counter-clockwise, then cut across to the inner loop, follow the loop, then cut back across to C, and follow the remainder of C. This curve represents the boundary of the region D, which does not contain 0, and so z^n is differentiable inside D and by Cauchy's Integral Formula, the integral over this curve is 0. However, this curve is made of four pieces (two of which cancel in the integral), so we are left with

$$0 = \oint_C z^n\,dz - \oint_{|z|=R} z^n\,dz$$

where the minus sign comes from doing the inner integral in a clockwise direction. We can then conclude these integrals have the same value.

The new curve that we've chosen to integrate over can be parameterized in a much simpler way, using

$$\gamma(t) = Re^{it}$$

for $t \in [0, 2\pi]$. Then, for $n < 0$, we have that

$$\begin{aligned}\oint_{|z|=R} z^n \, dz &= \int_0^{2\pi} (Re^{it})^n (iRe^{it}) \, dt \\ &= iR^{n+1} \int_0^{2\pi} e^{(n+1)it} \, dt\end{aligned}$$

When $n \neq -1$, we have that

$$\begin{aligned}&= iR^{n+1} \left(\frac{e^{(n+1)it}}{i(n+1)} \Big|_0^{2\pi} \right) \\ &= 0\end{aligned}$$

since $e^{2\pi(n+1)i} = e^0 = 1$. When $n = -1$, the integrand becomes

$$\begin{aligned}&= iR^{(-1)+1} \int_0^{2\pi} e^0 \, dt \\ &= i \int_0^{2\pi} 1 \, dt \\ &= 2\pi i\end{aligned}$$

and so we conclude that

$$\boxed{\oint_C z^n \, dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1 \end{cases}}$$

Another way to see this is that z^n has an antiderivative for all $n \neq -1$, and so we can use the fundamental theorem of complex line integrals, but for $n = -1$, the antiderivative *should* be the logarithm, but this is not an honest function as previously discussed. Note that this result also does not depend on R!

This has a very interesting consequence: if $f(z)$ has a Laurent series centered at 0 (or for any z_0, but for simplicity we will center it at 0), then for some contour C surrounding 0 on which $f(z)$ is differentiable throughout the interior of the region (with the possible exception of at 0), we have that

$$\begin{aligned}\oint_C f(z) \, dz &= \oint_C \sum_{n=-\infty}^{\infty} a_n z^n \, dz \\ &= \sum_{n=-\infty}^{\infty} \oint_C a_n z^n \, dz \\ &= 2\pi i a_{-1}\end{aligned}$$

since all other terms in the sum will be 0. This value a_{-1} essentially captures the behavior of integrating $f(z)$ around 0, and is called the **residue** of $f(z)$ at 0, denoted $\operatorname{Res}(f, 0)$. At all the points where $f(z)$ is differentiable, a Taylor series will exist, so $a_{-1} = 0$ at those points, and so a residue at a point will only be nonzero when there is an isolated singularity at that point.

The easiest way to compute a residue is by extracting it from the Laurent series. For example, for a simple pole, we have that

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} (z - z_0) \left(\frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + \cdots \right)$$

$$= \lim_{z \to z_0} a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots$$
$$= a_{-1} = \text{Res}(f, z_0)$$

which allows us to compute the residue easily. If $f(z)$ has a pole of order n, we can use the formula

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)$$

which you can check does indeed pick out the correct coefficient of the Laurent series. For essential singularities, the computation is often more difficult and may require some series manipulation to extract.

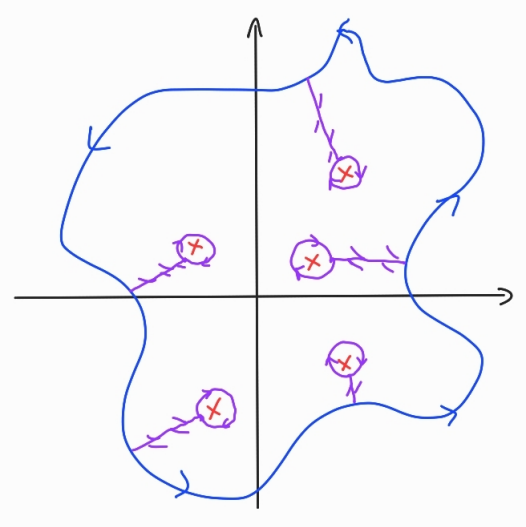


Figure 74: Reducing the integral around the blue contour to the sum of the integrals around each of the purple contours, where the singularities are indicated in red.

If $f(z)$ has multiple singularities inside of the contour C, then we can use a similar trick to reduce the integral around C to arbitrarily small loop integrals around each singularity. This gives us the result that

$$\oint_C f(z)\, dz = 2\pi i \sum_{\substack{z_i \text{ is an} \\ \text{isolated} \\ \text{singularity}}} \text{Res}(f, z_i)$$

which is known as the **Residue Theorem**. This theorem is incredibly powerful, as it allows us to compute integrals of functions which have singularities by simply summing over the singularities.

Example. Compute

$$\oint_{|z|=2} \frac{z+2}{z(z-1)}\, dz$$

We see that our integrand has two singularities, both simple poles, at $z = 0$ and $z = 1$. We compute

$$\text{Res}\left(\frac{z+2}{z(z-1)}, 0\right) = \lim_{z \to 0} \frac{z(z+2)}{z(z-1)} = \lim_{z \to 0} \frac{z+2}{z-1} = -2$$
$$\text{Res}\left(\frac{z+2}{z(z-1)}, 1\right) = \lim_{z \to 1} \frac{(z-1)(z+2)}{z(z-1)} = \lim_{z \to 1} \frac{z+2}{z} = 3$$

We thus conclude that

$$\oint_{|z|=2} \frac{z+2}{z(z-1)}\, dz = 2\pi i(3-2) = \boxed{2\pi i}$$

We are now ready to do a very cool integral! The integral, although entirely real, relies heavily on complex integration methods.

Example. Compute

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx$$

The first observation we make is that, when x is a real number, we have

$$\frac{\cos(x)}{x^2+1} = \operatorname{Re}\left(\frac{e^{ix}}{x^2+1}\right)$$

since the denominator is real. We then have that

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\,dx = \operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+1}\,dz$$

and so let

$$I = \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2+1}\,dz$$

We then consider the integrals over the following contours.

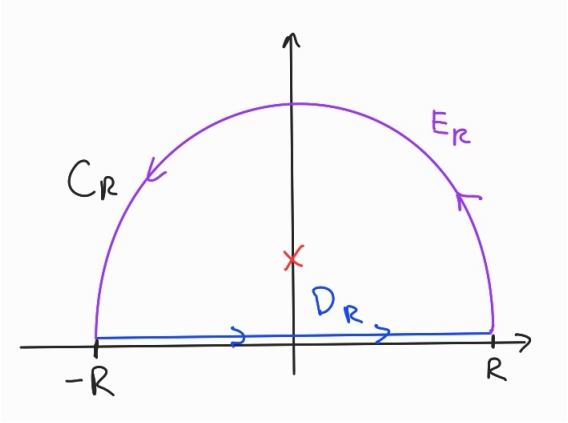


Figure 75: For each value of R, we define the contours D_R and E_R, which together form the loop C_R.

We have that

$$\oint_{C_R} \frac{e^{iz}}{z^2+1}\,dz = \oint_{D_R} \frac{e^{iz}}{z^2+1}\,dz + \oint_{E_R} \frac{e^{iz}}{z^2+1}\,dz$$

and note that

$$I = \lim_{R\to\infty} \oint_{D_R} \frac{e^{iz}}{z^2+1}\,dz$$

is the desired integral. We compute the integral over E_R using the parameterization $z(t) = Re^{it}$ for $t \in [0,\pi]$ as follows.

$$\begin{aligned}\oint_{E_R} \frac{e^{iz}}{z^2+1}\,dz &= \int_0^{\pi} \frac{e^{i(Re^{it})}}{(Re^{it})^2+1}(iRe^{it})\,dt \\ &= \int_0^{\pi} \frac{iRe^{-R\sin(t)+iR\cos(t)}}{R^2e^{2it}+1}e^{it}\,dt\end{aligned}$$

We can bound the norm of the value of the integral by the product of the maximum norm of the integrand, times the length of the contour. This is known as the **ML-estimate**, where M stands for "maximum" and L stands for "length". Thus, we have

$$\left|\int_0^\pi \frac{iRe^{-R\sin(t)+iR\cos(t)}}{R^2e^{2it}+1}e^{it}\,dt\right| \leq \pi \left|\frac{iRe^{-R\sin(t)}e^{iR\cos(t)}}{R^2e^{2it}+1}e^{it}\right|_{\substack{\text{for some } t\in[0,\pi]\\ \text{maximizing the norm}}}$$
$$= \pi\left|\frac{R}{R^2-1}\right|e^{-R\sin(t)}$$

as terms of the form $e^{if(t)}$ have norm 1. Since $\sin(t) \geq 0$ for all $t \in [0, \pi]$, the exponent of $e^{-R\sin(t)}$ is less than 0, so this term is bounded by 1. We therefore have

$$\leq \pi\left|\frac{R}{R^2-1}\right|$$

and we see that as $R \to \infty$, this term goes to 0. We thus conclude that as $R \to \infty$, the integral over E_R goes to 0, so we have

$$\lim_{R\to\infty}\oint_{C_R}\frac{e^{iz}}{z^2+1}\,dz = \lim_{R\to\infty}\oint_{D_R}\frac{e^{iz}}{z^2+1}\,dz = I$$

However, the loop integral can be computed with complex methods, as the function z^2+1 factors as $(z+i)(z-i)$, which give us two simple poles at $z = \pm i$. Since C_R never contains $z = -i$, and contains $z = i$ when $R > 1$, the integral over C_R will be the same for any $R > 1$. Thus, we can compute the integral in the limit by computing the integral for any $R > 1$. We then have that

$$\oint_{C_R}\frac{e^{iz}}{z^2+1}\,dz = 2\pi i\,\mathrm{Res}\left(\frac{e^{iz}}{z^2+1}, i\right)$$

We can find this residue by computing

$$\lim_{z\to i}(z-i)\frac{e^{iz}}{z^2+1} = \lim_{z\to i}\frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i} = \frac{1}{2ie}$$

We then have that

$$I = \oint_{C_R}\frac{e^{iz}}{z^2+1}\,dz = 2\pi i\frac{1}{2ie} = \frac{\pi}{e}$$

Since this answer is completely real, we have that

$$\int_{-\infty}^{\infty}\frac{\cos(x)}{x^2+1}\,dx = \mathrm{Re}\left(\int_{-\infty}^{\infty}\frac{e^{iz}}{z^2+1}\,dz\right) = \mathrm{Re}(I) = \boxed{\frac{\pi}{e}}$$

which is a pretty cool value for an integral.