Inverse Problems REU Notes

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These are notes for the Inverse Problems REU held at the UW during Summer 2014. They are updated online at:

http://www.math.washington.edu/~avius
http://www.math.washington.edu/~reu/papers/current/

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Inverse Problems REU Notes by Avi Levy

Lecture 1: June 23

Continuous Electrical Conductivity Problem

The mathematics of a flow of electrical current in a flate plate $W \subset \mathbb{R}^2$.

Definition 1.1. The **current flow density** is $\vec{J}(x,y)$, a vector field at each point of the plate that points in the direction the current flows.

Consider a curve C in the plate. At each point of C, there is a normal vector $\vec{n}$.

Definition 1.2. The **current** that flows across C in the direction $\vec{n}$ is defined to be

$$\int_C (\vec{J} \cdot \vec{n})\, ds.$$

For example, if the density is constant then the current across a line segment is length $\times$ density. Think of $\vec{J}$ as a given, then we use it to compute other quantities.

Definition 1.3. The **boundary** of a region W is called ∂W, and it is oriented such that W lies to the left of ∂W.

We will assume that ∂W is "nice", by which we mean that it is piecewise-smooth; there are only finitely many "corners".

Since we're in the plane, we can think in terms of complex numbers. Given a tangent vector

$$\vec{T} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right) = \frac{dx}{ds} + i\frac{dy}{ds},$$

we can multiply by $-i$ to rotate it to the right. This yields the outward pointing normal vector

$$\vec{n} = -i\vec{T} = \frac{dy}{ds} - i\frac{dx}{ds}.$$

Definition 1.4. A region is called **relatively compact** if it is bounded.

We will only work with bounded regions, to avoid issues with integration over infinite regions. This corresponds to the finite graphs that we will consider.

The total flow across ∂W is

$$\int_{\partial W} (\vec{J} \cdot \vec{n})\, ds.$$

How can we mathematically describe "inward-pointing"? If we move in the direction of $\vec{v}$ from p, then we will be at $p + t\vec{v}$ for some small $t > 0$. Then pointing inward means that $p + t\vec{v} \in W$ for small enough $t > 0$.

Let's think of a little disk $D_\varepsilon \subset W$. For instance, take a point $p \in W$ and let D_ε be the following set:

$$D_\varepsilon = \{p : |q - p| < \varepsilon\}.$$

Definition 1.5. The **net flow out of** D_ε is given by

$$\int_{\partial D_\varepsilon} (\vec{J} \cdot \vec{n}) \, ds.$$

Our convention is that outward is positive and inward is negative.

Suppose that there are no sources or sinks of current. Mathematically, this means that

$$\int_{\partial D_\varepsilon} (\vec{J} \cdot \vec{n}) \, ds = 0,$$

for all disks D_ε in W.

Now we apply a version of Green's Theorem, which usually goes by the name of the **Divergence Theorem**. This says that if $J = (P, Q)$, then

$$\int_{\partial D_\varepsilon} (\vec{J} \cdot \vec{n}) \, ds = \int_{D_\varepsilon} (P_x + Q_y) \, dA.$$

We will look at analogues of this identity for graphs.

Theorem 1.6 (Green's Theorem)**.** *Consider a vector field $\vec{F} = (P, Q)$ on a region W with a smooth boundary ∂W. Then*

$$\begin{aligned}\int_{\partial W} (\vec{F} \cdot \vec{T}) \, ds &= \int_{\partial W} (P \, dx + Q \, dy) \\ &= \int_W (Q_x - P_y) \, dx \, dy.\end{aligned}$$

Recall that $Q_x = \frac{\partial Q}{\partial x}$ and likewise for P_y.

Let's recall what this means in terms of our earlier calculations of $\vec{T}$ and $\vec{n}$. Then

$$\begin{aligned}\int_{\partial W} \vec{J} \cdot \vec{n} \, ds &= \int_{\partial W} \left(P \frac{dy}{ds} - Q \frac{dx}{ds} \right) ds \\ &= \int_{\partial W} P \, dy - Q \, dx \\ &= \int_{\partial W} (-Q \, dx + P \, dy) \\ &= \int_W (P_x + Q_y) \, dx \, dy.\end{aligned}$$

Note: For those of you familiar with differential forms, there is another way to see this. The differential form $\omega = -Q \, dx + P \, dy$ has exterior derivative

$$d\omega = -Q_y \, dy \wedge dx + P_x \, dx \wedge dy = (Q_y + P_x) \, dx \wedge dy.$$

Then the identity we just derived reads

$$\int_{\partial W} \omega = \int_W d\omega,$$

which follows directly from the generalized Stokes' Theorem.

Return from break

We're not going to worry about all the conditions, like the amount of differentiability of our functions; in other words, all our functions are infinitely differentiable. Now if

$$\int_{D_\varepsilon} f \, dA = 0$$

for every disk, and f is continuous, it follows that $f = 0$. Indeed if f didn't vanish identically, then there would be some point $f(p) > 0$ (or $f(p) < 0$, but this case is similar). Then by continuity of f, there is a disk D_ε on which $f > 0$. It follows that $\int_{D_\varepsilon} f > 0$, which is a contradiction. Thus f vanishes identicially.

Apply this to $f = \operatorname{div}(\vec{J})$; it follows that the condition for "no sources or sinks of current" is equivalent to saying $\operatorname{div}(\vec{J}) = 0$ identically.

Now we're going to make two physical approximations, in order to study the problem. Consider an electric field $\vec{E}$ that induces our current, and suppose that $\vec{J} = \gamma \vec{E}$ **(assumption 1)**. Then γ depends on the material the plate is made out of. We will not assume our plate is homogeneous, which means that γ depends on its location. In other words,

$$\vec{J}(p) = \gamma(p) \vec{E}(p).$$

We will assume that γ is a positive infinitely differentiable function. Our statement is that $\operatorname{div}(\gamma \vec{E}) = 0$.

Next, we suppose that $\vec{E}$ comes from a potential function u **(assumption 2)**. For physicists, this means

$$\begin{aligned} \vec{E} &= -\operatorname{grad}(u) \\ &= -\nabla u \\ &= -(u_x, u_y). \end{aligned}$$

Plugging in to the earlier equation, we have

$$\begin{aligned} 0 &= \operatorname{div}(\gamma \vec{E}) \\ &= \operatorname{div}(\gamma(-\operatorname{grad} u)). \end{aligned}$$

This is an extremely important equation, and we write it as

$$\boxed{\nabla \cdot (\gamma \nabla u) = 0}$$

In terms of components, we have $\nabla = (\partial_x, \partial_y)$. Therefore

$$\partial_x[\gamma u_x] + \partial_y[\gamma u_y] = \gamma_x u_x + \gamma_y u_y + \gamma(u_{xx} + u_{yy}) = 0.$$

We can rewrite this as

$$(\nabla\gamma \cdot \nabla u) + \gamma(\nabla^2 u) = 0.$$

When γ is constant, the first term drops out and the equation becomes $\nabla^2 u = 0$; this is called the **Laplace Equation**.

Definition 1.7. A solution, u, to the Laplace equation is called a **harmonic function**.

Harmonic functions are very well studied. We will work with analogues of these functions:

Definition 1.8. A solution, u, to the equation $\nabla \cdot (\gamma \nabla u) = 0$ is called a **γ-harmonic function**.

We will actually work with "analogues of the analogues", by considering γ-harmonic functions on graphs.

Well-posed problems

A large part of mathematics is about coming up with the right problem to work on.

Definition 1.9. A problem is called **well-posed** if:

1. Solutions exist
2. Solutions are unique
3. The solution depends continuously on data
4. There is an algorithm to find the solution

Historically, it was Calderon (1979) who first asked the question that we will work with. Consider the operator L given by

$$Lu = \operatorname{div}(\gamma \operatorname{grad} u) = \nabla \cdot (\gamma \nabla u)$$

This operator is very special; it is a linear second order operator, that happens to be homogeneous and **elliptic**. We won't really define elliptic yet, but it is really important.

The **Calderon Problem** is to solve $Lu = 0$ on W, with $u = \phi$ on ∂W. There are lots of conditions we won't get into, for instance the limit of u has to approach ϕ on the boundary. Well-posedness was very difficult to establish, and many famous mathematicians worked very hard to solve it.

Recall the normal derivative

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n}.$$

Then we consider the quantity $\gamma\frac{\partial u}{\partial n} = \gamma\nabla u \cdot \vec{n}$, which is the normal component of the flow density. We call this quantity ψ; that is,

$$\psi = \gamma\frac{\partial u}{\partial n}.$$

Definition 1.10. The mapping $\phi \mapsto \psi$ is called the **Dirichlet-to-Neumann** map, and denoted by Λ. We also call it the **response map**. We refer to $\Lambda(\phi)$ by writing Λ_ϕ.

We are interested in the following question:

Does Λ determine γ?

This is the **inverse problem** we are concerned with.

Note that the map Λ is really complicated, especially in the continuous case. In general, it cannot be written down; it is not as nice as the map L, which is a relatively simple differential operator; just take some derivatives and multiply them as necessary.

Summary

We motivated an equation, by saying it describes if there are sources or sinks. Then we asked what we can say about solving that equation with given the given boundary data. From that, we can produce the map Λ.

Now we suppose that Λ is given; we want to determine information about the interior of the region, i.e. we want to determine γ.

Prove the following equation during lunch:

$$\int_{\partial W}\left[u\gamma\frac{\partial v}{dn} - v\gamma\frac{\partial u}{\partial n}\right] ds = \int_W (uLv - vLu).$$

The last 40 minutes

Couple of important identities

$$\int_{\partial W} u\gamma\frac{\partial v}{\partial n} = \int_W \gamma\nabla u\cdot\nabla v + \int_W uLv \tag{1}$$

$$\int_{\partial W} u\gamma\frac{\partial u}{\partial n} = \int_W \gamma|\nabla u|^2 + \int_W uLu \tag{2}$$

$$\int_{\partial W} u\gamma\frac{\partial u}{\partial n} = \int_W \gamma|\nabla u|^2 + 0. \tag{3}$$

This leads to

$$\int_{\partial W}\phi\Lambda_\phi = \int_W \gamma|\nabla u|^2.$$

Dirichlet innerproduct of norm

Define the inner product (a bilinear form)

$$\langle u, v\rangle = \int_W \gamma \nabla u \cdot \nabla v$$

and introduce the norm $\|u\|^2 = \int_W \gamma |\nabla u|^2 \geq 0$.

Theorem 1.11. *Let $T = \{w\colon w = \phi \text{ on } \partial W\}$. Suppose w_0 minimizes $\{\|w\|^2 : w \in T\}$ then $\langle w_0, f\rangle = 0$ for all $f = 0$ on ∂W and conversely if $w_0 = \phi$ on ∂W and $\langle w_0, f\rangle = 0$ for all f such that $f = 0$ on ∂W then $\|w_0\|^2$ is minimal for al lw such that $w = \phi$ on ∂W.*

Proof.

($\Rightarrow$) Define the function $q(t) = \|w_0 + tf\|^2$. We compute

$$\begin{aligned} q(t) &= \langle w_0 + tf, w_0 + tf\rangle \\ &= \langle w_0, w_0\rangle + 2t\langle w_0, f\rangle + t^2\langle f, f\rangle \\ &= \|x_0\|^2 + 2t\langle w_0, f\rangle + t^2\|f\|^2. \end{aligned}$$

Since w_0 is a minimum we have $q'(t) = 2\langle w_0, f\rangle + 2t\|f\|^2$. So, if $q'(0) = 0$ then $\langle w_0, f\rangle = 0$. Note we need $f \equiv 0$ on ∂W otherwise $w_0 + tf \notin T$.

($\Leftarrow$) Suppose that $\langle w_0, f\rangle = 0$ for all $f \equiv 0$ on ∂W. Take another w with $\|w\|^2 = q(t)$. The same calculation gives

$$\|w\|^2 = q(t) = \|x_0\|^2 + 2t\langle w_0, f\rangle + t^2\|f\|^2 = \|w_0\|^2 + t^2\|f\|^2 \geq \|w_0\|^2.$$

By assumption this is equal to $\|x_0\|^2 + t^2\|f\|^2$ □

Solving the Dirichlet problem

Suppose w_0 minimizes $\|w_0\|^2$ among all w with $w \equiv 0$ on ∂W. By the previous theorem, this is true if and only if $\langle w_0, f\rangle = 0$ for all $f = 0$ on ∂W. So we have

$$\int_W \gamma \nabla w_0 \nabla f = 0.$$

We make use of (1) to write as $\int_W f(Lw_0) = 0$ for all $f \equiv 0$ on ∂W. Exercise: prove that $Lw_0 = 0$.

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Lecture 2: June 24

Identities kept "permanently" on the board:

$$\int_{\partial W} v\gamma\frac{\partial u}{\partial n}\,ds = \int_W vLu\,dA + \int_W \gamma(\nabla v\cdot\nabla u)\,dA$$

$$\int_{\partial W} u\gamma\frac{\partial u}{\partial n}\,ds = \int_W uLu\,dA + \int_W \gamma|\nabla u|^2\,dA$$

$$\langle u,v\rangle_\gamma = \int_W \gamma\nabla u\cdot\nabla v \qquad (\star)$$

$$\|u\|_\gamma^2 = \langle u,u\rangle_\gamma = \int_W \gamma|\nabla u|^2.$$

Equation $(\star)$ is the definition of something that is "nearly" an inner product. It's only defect is that $\|u\|_\gamma = 0$ does not necessarily imply $u \equiv 0$; indeed, u could be any constant function.

Solution to the homework problem

Suppose $\int_W f\, Lw_0\, dA = 0$ whenever $f = 0$ on ∂W. Show that $Lw_0 \equiv 0$.

Proof. Suppose to the contrary that $Lw_0 \not\equiv 0$. Then there is some $p \in W$ such that $Lw_0(p) \neq 0$. Consider the case $Lw_0(p) > 0$ (the case $Lw_0(p) < 0$ is similar). Recall that Lw_0 is a continuous function on W; therefore there is a small disk D_ε around p such that $Lw_0(q) > 0$ for all $q \in D_\varepsilon$.

Now we **choose** a special function f; it will be a function that is 0 everywhere outside of D_ε, and positive everywhere inside of D_ε. We can make f as smooth as we want (i.e., be continuous with lots of derivatives).

With this choice of f, we compute that

$$\int_W f\, Lw_0\, dA = \int_{D_\varepsilon} f\, Lw_0 > 0,$$

since the expressions latter integral are strictly positive inside of D_ε. This contradicts our hypothesis that $\int_W f\, Lw_0\, dA = 0$, so we have proven the result. □

Consider two functions ϕ, ψ on ∂W. Define a new inner product

$$\langle\phi,\psi\rangle = \int_{\partial W} (\phi\psi)\,ds.$$

This is really nice; it has no defects like the previous inner product (but it only works on the boundary). Any time you have an inner product, you have a norm as well; the present norm is given by

$$\|\phi\|^2 = \int_{\partial W} \phi^2 \, ds.$$

Plugging into this inner product, we obtain

$$\int_{\partial W} \psi \Lambda \phi = \langle \psi, \Lambda \phi \rangle$$

With this inner product, we can start forming analogies between Λ and bilinear forms.

Definition 2.1. A matrix A is called **positive semi-definite** if $x^T A x \geq 0$ for all vectors x.

Recall that if A is a positive semi-definite symmetric matrix, there is an associated bilinar form

$$\langle y, x \rangle = y^T A x.$$

We need symmetry of A in order to ensure that $\langle y, x \rangle = \langle x, y \rangle$, and positive semi-definiteness to ensure that $\langle x, x \rangle \geq 0$.

To build the analogy between Λ an matrices, observe that Λ is linear; this means

$$\Lambda(a\phi + b\psi) = a\Lambda\phi + b\Lambda\psi.$$

In addition, we will see that $\Lambda^T = \Lambda$ (symmetry) and Λ is positive semi-definite.

Given a bilinear form $b(x, y)$, there is an associated quadratic form $q(x)$ that is obtained by setting $q(x) = b(x, x)$. It turns out that given q, we can go backwards to find b. Start by computing

$$\begin{aligned} q(x+y) &= b(x+y, x+y) \\ &= b(x, x+y) + b(y, x+y) \\ &= b(x,x) + b(x,y) + b(y,x) + b(y,y) \\ &= q(x) + 2b(x,y) + q(y). \end{aligned}$$

Solving for $b(x, y)$, we obtain

$$b(x,y) = \frac{q(x) - q(y) - q(x+y)}{2}.$$

Return from break

Properties of Λ checklist:

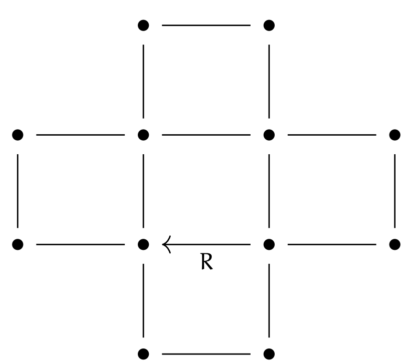
1. $\Lambda^T = \Lambda$

2. Λ is positive semi-definite

3. $\ker \Lambda$?

1. Consider a matrix $A: V \to V$. Saying a matrix is symmetric actually depends on your choice of inner product. Once you've picked an inner product, the definition of symmetry is that
$$\langle \psi, A\phi \rangle = \langle A\psi, \phi \rangle$$
for all vectors ψ, ϕ.
We can generalize this to the continuous case, using the inner product $\langle \psi, \phi \rangle = \int_{\partial W} \psi\phi \, ds$. Thus to show Λ is symmetric boils down to checking
$$\langle \psi, \Lambda\phi \rangle = \langle \Lambda\psi, \phi \rangle .$$
Using $\Lambda\phi = \gamma \left.\frac{\partial u}{\partial n}\right|_{\partial W}$, this reduces to the equality
$$\int_{\partial W} \psi\gamma \frac{\partial u}{\partial n} = \int_{\partial W} \phi\gamma \frac{\partial v}{\partial n},$$
where $u = \phi$ on ∂W and $v = \psi$ on ∂W and $Lu = Lv = 0$. However by identity 1, we obtain
$$\int_{\partial W} \psi\gamma \frac{\partial u}{\partial n} = \int_{W} \gamma(\nabla v \cdot \nabla u) \, dA = \int_{\partial W} \phi\gamma \frac{\partial v}{\partial n}.$$
2. Applying identity 2, we see that
$$\langle \psi, \Lambda\psi \rangle = \int_W \gamma |\nabla u|^2 \, dA \geq 0.$$
Here we have used positivity of γ.
3. First, suppose that $\phi \in \ker \Lambda$. Then $\langle \phi, \Lambda\phi \rangle = 0$, so by identity 2
$$\int_W \gamma |\nabla u|^2 \, dA \implies \nabla u \equiv 0.$$
Thus u is locally constant, so it is constant on each component of W. Similarly, all such functions belong to the kernel. Hence the kernel is a vector space whose dimension is the number of connected components of W.

Preview of electrical networks



An electrical network is a graph with wires connecting its nodes. Along each wire, there is a resistance that causes a voltage drop. We will work with the reciprocal of the resistance, which is called the **conductivity**. We think of the network as a black box; we can only ineract with the boundary, and we will try to determine the inside.

Return from lunch

Definition 2.2. A **graph** is a pair (V, E) where V is the set of vertices and E is the set of edges.

We will only talk about finite graphs, but studying infinite graphs is also a fruitful area for research. When V is finite, we write $V = \{v_1, \cdots, v_n\}$ and $|V| = n$ (the **cardinality** of V).

For simplicity, we start with a fairly restrictive definition of a graph. Our graphs will have no self-loops, no multiple edges, and will be undirected (sometimes called a **simple** graph). These requirements are all related to the **edge set** E, which we think of as a subset of $V \times V$.

From this perspective, an edge $e \in E$ is thought of as an ordered pair $e = (p, q)$, with $p, q \in V$. Already, we're getting a bit tongue-tied; the problem is that (p, q) and (q, p) denote different elements of $V \times V$, so the current definition leads to a **directed** graph.

To get around this problem, we need to define a **quotient set** $\widetilde{V \times V} = (V \times V)/\sim$, where $\sim$ is the relation that says $(p, q) \sim (q, p)$. This relation means that we are working with unordered pairs. Then the subsets of $\widetilde{V \times V}$ correspond to the edge sets of **undirected** graphs.

Thus we've made our edges undirected. To remove self-loops, we have to disallow the edges of the form (p, p). We refer to the set

$$D = \{(p, p) : p \in V\} \subset V \times V$$

as the **diagonal** subset of $V \times V$. Thus the complete definition of a simple graph is

$$G = (V, E) \qquad E \subset \widetilde{V \times V} \setminus D$$

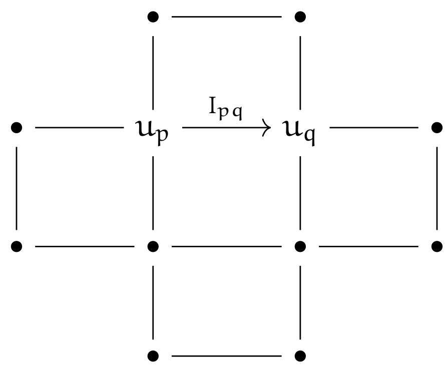
In practice, we will refer to edges as pairs (p, q) with the convention that the order doesn't matter. Furthermore, if we have enumerated our vertices as $\{v_i\}_{i=1}^n$, we will refer to the edge (v_i, v_j) as simply (i, j).

Definition 2.3. An **electrical network** Γ is a graph with conductivities:

$$\Gamma = (G, \gamma)$$

Thue function $\gamma : E \to \mathbb{R}_{>0}$ assigns a (strictly) positive real number to each edge e. We refer to $\gamma(e)$ as the conductivity of the edge e, and physically it corresponds to the reciprocal of the resistance. If $e = (p, q)$, we refer to $\gamma(e)$ as γ_{pq}.

Suppose we have a potential function $u\colon V \to \mathbb{R}$. Then Ohm's Law boils down to saying that $I_{pq} = \gamma_{pq}(u_p - u_q)$ is the current flowing from p to q.



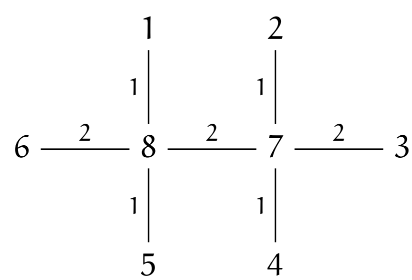
The Kirchhoff Matrix

Even though the graphs we encounter may be very complicated, we can always encoud their information into a single 2×2 matrix.

Definition 2.4. The **Kirchhoff matrix** is the matrix $K = (\kappa_{ij})_{i,j=1}^{n}$ given by

$$\kappa_{ij} = \begin{cases} -\gamma_{ij} & i \neq j \\ \sum_{k \neq i} \gamma_{ik} & i = j \end{cases}$$

Note that $\kappa_{ij} = \kappa_{ji}$, so K is always a symmetric matrix.



The Kirchhoff matrix for this graph is as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & & & & & & & 0 \\ 0 & & \ddots & & & & & 0 \\ 0 & & & \ddots & & & & 0 \\ 0 & & & & \ddots & & & -1 \\ 0 & & & & & \ddots & & -2 \\ 0 & & & & & & & -2 \\ -1 & 0 & 0 & 0 & -1 & -2 & -2 & 6 \end{bmatrix}$$

Consider a vector x. We compute Kx by plugging into the definitions:

$$
\begin{aligned}
(Kx)_i &= \kappa_{ii} x_i + \sum_{j \neq i} \kappa_{ij} x_j \\
&= \left(\sum_{j \neq i} \gamma_{ij} \right) x_i - \sum_{j \neq i} \gamma_{ij} x_j .
\end{aligned}
$$

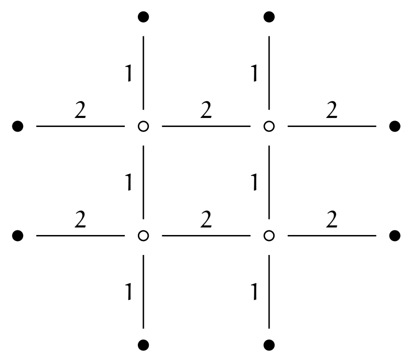
We introduce the notation $j \sim i$, which means that i, j are adjacent; i.e. (i, j) is an edge. Continuing,

$$
\begin{aligned}
(Kx)_i &= \left(\sum_{j \neq i} \gamma_{ij} \right) x_i - \sum_{j \neq i} \gamma_{ij} x_j \\
&= \sum_{j \sim i} \gamma_{ij} x_i - \sum_{j \sim i} \gamma_{ij} x_j \\
&= \sum_{j \sim i} \gamma_{ij} (x_i - x_j)
\end{aligned}
$$

Inverse Problems REU Notes by Avi Levy

Lecture 3: June 25

Some notation: boundary/interior nodes denoted by $\bullet = \partial V$ and $\circ = \operatorname{Int} V$



We may write the Kirchhoff matrix for this network in the following form:

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = (\kappa_{ij})$$

The entries A, B, C are **block entries**, and they are matrices themselves. The first $|\partial V|$ entries correspond to the boundary nodes, and the last $|\operatorname{Int} V|$ entries correspond to the interior nodes. We write ϕ for the column vector containing the interior nodes, and we write x for the vector containing the boundary nodes. When we write the matrix in this form, we obtain the solution to the Dirichlet problem:

$$x = -C^{-1}B^T\phi.$$

Indeed, solving the Dirichlet corresponds to the equation

$$K\begin{pmatrix} \phi \\ x \end{pmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}\begin{pmatrix} \phi \\ x \end{pmatrix} = 0.$$

Multiplying the matrices yields $B^T\phi + Cx = 0$, from which we obtain $x = -C^{-1}B^T$ as stated.

We can row-reduce this matrix to obtain the block matrix $\Lambda = A - BC^{-1}B^T$. The response map is given by

$$\phi \mapsto \Lambda\phi = (A - BC^{-1}B^T)\phi.$$

Note that λ_{ij} is the current at vertex i due to a voltage of 1 at j and 0 elsewhere.

Theorem 3.1. *Properties of* $\Lambda = (\lambda_{ij})$*:*

1. $\Lambda^T = \Lambda$
2. $\phi^T\Lambda\phi \geq 0$

3. $\ker\Lambda = \{$*constant* ∂*-values*$\}$

4. $\lambda_{ij} \le 0$, *and* $\lambda_{ij} < 0$ *if* i, j *are connected by a path through* $\operatorname{Int} V$

The problem: given a ϕ on ∂V, is there an x on $\operatorname{Int} V$ such that the net current flow at each interior vertex is 0?

If i is adjacent to j we write $i \sim j$. The following quantity is the net current flow at vertex i:

$$I_i(v) = \sum_{j\sim i} \gamma_{ij}(v_i - v_j) = (Kv)_i.$$

Lastly, there is an identity

$$y^T Kx = \frac{1}{2}\sum_{i,j} \gamma_{ij}(x_i - x_j)(y_i - y_j).$$

C is positive definite, whereas K is not invertible.

Proof of λ properties. 1. Observe that A, C are both symmetric, so therefore C^{-1} is symmetric. Hence we compute

$$\begin{aligned}\Lambda^T &= (A - BC^{-1}B^T)^T \\ &= A^T - (B^T)^T(BC^{-1})^T \\ &= A - B(C^{-1})^T B^T \\ &= A - BC^{-1}B^T = \Lambda.\end{aligned}$$

Therefore Λ is symmetric.

2. Write $u = \begin{pmatrix}\phi \\ x\end{pmatrix}$. By grouping interior and boundary edges in the sum for $u^T Ku$, we see that

$$\begin{aligned}u^T Ku &= \sum_{\partial V} u_i(Ku)_i + \sum_{\operatorname{Int} V} u_i(Ku)_i \\ &= \sum_{\partial V} u_i(Ku)_i \\ &= \sum_{\partial V} u_i I_i(u) \\ &= \sum_{\partial V} \phi_i(\Lambda\phi)_i \\ &= \phi^T\Lambda\phi.\end{aligned}$$

But we know that

$$u^T Ku = \sum_{i<j} \gamma_{ij}(u_i - u_j)^2 \ge 0,$$

so therefore $\phi^T\Lambda\phi \ge 0$ as desired.

3. Suppose $\Lambda\phi = 0$. Then $\phi^T\Lambda\phi = 0$, so by the previous part $u^T K u = 0$. Therefore

$$\sum_{i\sim j} \gamma_{ij}(u_i - u_j)^2 = 0,$$

so $u_i = u_j$ whenever $i \sim j$. Since the graph is connected, this implies that u is constant, so therefore ϕ is constant as well.

4. Suppose that u is γ-harmonic. Then if $i \in \operatorname{Int} V$, we know that

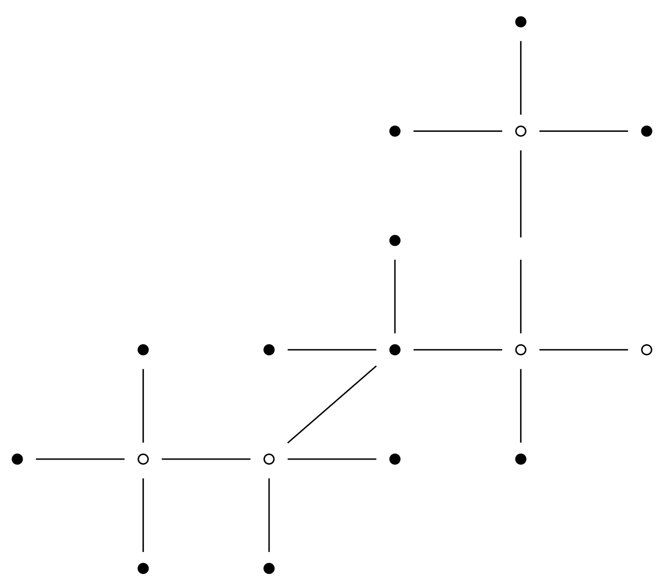
$$\sum_{j\sim i} \gamma_{ij}(u_i - u_j) = 0.$$

It follows that if u_i is a local maximum of u, then $u_i = u_j$ for all j adjacent to i; in other words, at any local maximum the function is locally constant. Note that this is not a global statement; later on we will construct an exotic graph that is 0 on half of its domain. For the rest of the proof, we will need to introduce the maximum principle.

□

Theorem 3.2 (The Maximum Principle)**.** *Let u be γ-harmonic. Then the maximum of u is attained on ∂V.*

Theorem 3.3 (Mean Value Property)**.** $u_i = \sum_{j\sim i} w_j u_j$, *where* $w_j = \frac{\gamma_{ij}}{\sigma}$ *and* $\sigma = \sum_{j\sim i} \gamma_{ij}$.



Return from lunch

Another proof of the existence and uniqueness of solutions to the Dirichlet property.

Theorem 3.4. *Suppose ϕ is given on ∂V. Then there is a unique u which is γ-harmonic and $u = \phi$ on ∂V.*

Proof. The condition for γ-harmonicity is that $(Ku)_i = 0$ for all $i \in \operatorname{Int} V$. Since the boundary values of u are known, we only have $\operatorname{Int} V$-many unknowns. Notice that the system of equations $(Ku)_i = 0$ is not homogeneous, since it incorporates constant terms coming from the ϕ's. Thus our system is of the form $Tx = b$.

To determine if this system has a unique solution, we need only verify that the equation $Tx = 0$ has a unique solution. But this statement follows from the following claim:

Claim 3.5. *If a γ-harmonic function u vanishes on ∂V, then $u \equiv 0$.*

Proof. Since u attains its maximum on the boundary, $\max u = 0$ (by the maximum principle). Recall that there is also a minimum principle, which similarly proves that $\min u = 0$. Thus $u \equiv 0$. □

□

See the graph from before lunch. The sign condition hinges on a certain topological property: there must be a way to connect v_i to v_j by a path through the interior. Our convention is that net current flow is out of the boundary vertex and into the rest of the graph. Since there are no sources or sinks on the interior of the graph, it follows that $\lambda_{ij} \leq 0$.

Definition 3.6. Two boundary vertices i, j are said to be **connected through** $\operatorname{Int} V$ if either i, j are adjacent, or if there is a path from i to j consisting entirely of points in $\operatorname{Int} V$.

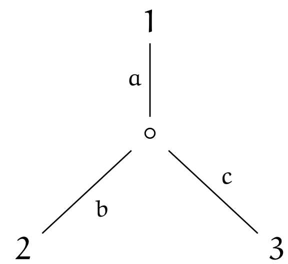
Claim 3.7. $\lambda_{ij} = 0$ *if and only if* i, j *are connected through* $\operatorname{Int} V$.

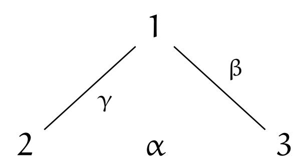
Keep the example graph in mind (to follow along with the proof).

Proof. Suppose that $\lambda_{ij} < 0$. The case $i = j$ is clear, so suppose that $i \neq j$. Place a boundary potential of 1 at j and 0 elsewhere; we refer to this situation by writing $\phi_i = \delta_{ij}$. Solve the Dirichlet problem to obtain u. Since $u_i = 0$, it follows that there is some neighbor of i with non-zero current flow towards i. If this neighbor is a boundary node, then we are done. Otherwise, we may continue this process until it terminates. Since the only way to terminate is to hit a boundary node, the process constructs a path through $\operatorname{Int} V$ connecting the boundary nodes.

The other direction is similar; if we have a path through $\operatorname{Int} V$, then (since none of the conductivities vanish) the current splits at each interior node into positive pieces. Thus there will be a positive current flow. □

$Y-\Delta$





The response matrix for the second graph is just the Kirchhoff matrix. Thus we obtain

$$\begin{bmatrix} \beta+\gamma & -\gamma & -\beta \\ -\gamma & \alpha+\gamma & -\alpha \\ -\beta & -\alpha & \alpha+\beta \end{bmatrix}$$

Now we compute the Kirchhoff matrix for the first graph. It follows that

$$K = \begin{bmatrix} * & 0 & 0 & -a \\ * & * & 0 & -b \\ * & * & * & -c \\ * & * & * & * \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & -a \\ 0 & b & 0 & -b \\ 0 & 0 & c & -c \\ -a & -b & -c & a+b+c \end{bmatrix}$$

Row-reducing using the bottom row yields

$$\Lambda = \frac{1}{a+b+c}\begin{bmatrix} a(b+c) & -ab & -ac \\ -ab & b(a+c) & -bc \\ -ac & -bc & c(a+b) \end{bmatrix}$$

Set $\sigma = a+b+c$. Then we obtain

$$\alpha = \frac{bc}{\sigma},\ \beta = \frac{ac}{\sigma},\ \gamma = \frac{ab}{\sigma}.$$

Therefore we compute

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{abc\sigma}{\sigma^2} = \frac{abc}{\sigma}.$$

Now since $\alpha = \frac{bc}{\sigma}$, we find that

$$a = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha}$$

and likewise for b, c. This is the simplest example of **network recovery**. This is a visualization of Gaussian elimination.

Schur Complements

The general situation we are concerned with is as follows:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, D are square matrices and D is invertible.

Definition 3.8. The **Schur complement** M/D is the top-left block matrix that results via block Gaussian elimination of M along the bottom row.

To perform the Guassian elimination, start by multiplying the bottom by BD^{-1} to obtain

$$(I_A\ BD^{-1})M = \begin{bmatrix} A & B \\ BD^{-1}C & BD^{-1}D \end{bmatrix} = \begin{bmatrix} A & B \\ BD^{-1}C & B \end{bmatrix}$$

Now subtract the bottom row from the top row to obtain

$$(I_M - I_A\ I_D)(I_A\ BD^{-1})M = \begin{bmatrix} A - BD^{-1}C & 0 \\ C & D \end{bmatrix}$$

Thus $M/D = A - BD^{-1}C$. Since we have only done row operations, the determinant is unchanged. Therefore

$$\det M = \det\left(\begin{bmatrix} M/D & 0 \\ C & D \end{bmatrix}\right) = \det(M/D)\det D.$$

Another way of saying this is that $\det(M/D) = \det M/\det D$. Next time we will discuss uniqueness of block Gaussian elimination.