Course notes for Manifolds

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These are my notes for Math 546, taught by Judith Arms at the UW in Spring 2014. They are updated online at:

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Lecture 1: March 31

Old Homework Talk

Recall problem 9-17(a). $V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, and find flow coordinates (s^1, s^2, s^3). One of her goals (maybe the only reason it was assigned) is to notice that $V_1 = \frac{\partial}{\partial x} = \frac{\partial}{\partial s^1}$ but $x \neq s^1$. In particular, $s^1 = x - 1 - y, s^2 = y, s^3 = z$. Observe that s^1 is a function that satisfies $ds^1(V_2) = ds^1(V_3) = 0$. Equivalently, V_2 and V_3 are tangent to the s^1 level sets. The majority of the information about s^1 are what the level sets are; the only thing that's missing is how fast the lines change, which is like a scaling. Later, we'll call the surfaces that are tangent the "integral curves of the distribution" given by the rest of the vector fields.

$ds^i(V_{j\neq i}) = 0$ determines the level sets of s^i, and $ds^i(V_i) = 1$ only determines "how fast s^1 changes between the level sets" (in quotes because it is a fairly heuristic remark).

Also related: $\{s^i \text{ coordinates}\} = \{\bigcap_{j\neq i} \text{ level sets of } s^j\}$.

New Homework Talk

The W problems will just be a whole bunch of easy calculations $(11 - 7, 9, 10, 14)$. Do what you need to do to be at ease with the calculations; not necessary to do absolutely everything if you are comfortable with what's going on.

Review:

1. $F: M \to N$ and $u: N \to \mathbb{R}$. Then $du_q: T_qN \to T_{u(q)}\mathbb{R} \simeq \mathbb{R}$, identify $du_q \in T_q^*N$ (we are expressing the latest thing in terms of something we knew before).

2. F_p^* is defined to be $(dF_p)^*$ (we are doing adjoints here). If $v \in T_pM$, then $(F^*du)_p(v) = du_{F(p)}(dF_p(v))$ (by definition of the adjoint map). Now we use the other nature of du; besides being a covector, it is also the differential of a function. So by chain rule, the last expression $= d(u \circ F)_p(v)$ (recall $du: T_{F(p)}N \to \mathbb{R}$). Hence $(F^*du)_p = d(u \circ F)_p$. Now recall from way back that $u \circ F = F^*u$ (see Problem 2-10). Now a preview of tensor manipulations (i.e. differential forms) is that our expressions for $(F^*du)_p(v)$ becomes $d(F^*u)_p(v)$, whereupon $F^*du = dF^*u$; something commutes with something. Also note linearity of dF_p (hence, linearity of F^*); this means $F^*(u\alpha + v\beta) = (u \circ F)F^*\alpha + (v \circ F)F^*(\beta)$. In other words,

$$F^*(u\alpha + v\beta) = F^*uF^*\alpha + F^*vF^*(\beta).$$

So F^* is acting on the algebra of covariant tensors, and it distributes over the tensor product; the tensor product of a rank-0 covector and a rank-1 covector distributes.

3. If $F = i\colon S \to N$ is an inclusion, and ω is a covector at p, then there is no ambiguity about $F^*\omega$ (like there was with vectors - this is due to asymmetry in definition of a function). This holds at every point $p \in S$. Compare with $dF(X)$ which may or may not be defined or multiply defined at points of N. Note: See the comment/example (in the book) on difference between $\omega|_S$ (section of the pullback bundle $T^N|_S$ given by $i\colon S \to N$; this is a covector over a map, compare to bundle homomorphism over a map). That is different than $i^*\omega\colon S \to T^*S$, which typically has smaller dimension (unless S is open). This is more complicated than pulling back a function, because a function is 0 or its not. This is tricky; it's because $\omega|_S$ is operating on a bigger vector space that $i^*\omega$. Again, there is a concrete $\mathbb{R}^n$ example worked out in the book.

Emphasis: For computation, "$\circ F$" just means plug in formula for F. Example is $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x,y) = (xy, x^2 - y^2) = (u,v)$. Then

$$\begin{aligned} F^*(u\ dv) &= (u \circ F)d(v \circ F) \\ &= (xy)d(x^2 - y^2) \\ &= (xy)(2x\ dx - 2y\ dy) \end{aligned}$$

If we think of F as a coordinate change, (we can do this locally in a neighborhood of any point where dF is non-singular - the origin), then $F^*\ dv = 2x\ dx - 2y\ dy$ and $F^*\ du = y\ dx + x\ dy$, and those are linearly independent except at the origin. Generically now: chance coordinates $(x^i) \to (u^i) = F(x^1, \cdots, x^n)$. Then $du^i = \frac{\partial u^i}{\partial x^j} dx^j$ (with summation convention). Tip: If you're thinking in index notation, you can't write down the wrong formula! (J. M. Arms, 2014). See also the analogous formula for coordinate vectors: $\frac{\partial}{\partial x^j} = \frac{\partial u^i}{\partial x^j}\frac{\partial}{\partial u^i}$.

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Lecture 2: April 2

Talking about the gradient

Exercise 11.17, differential of f, gradient of f. In Cartesian,

$$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

For $n = 2$, convert V to polar coordinates.

$$\begin{aligned} V &= V(r)\frac{\partial}{\partial r} + V(\theta)\frac{\partial}{\partial \theta} \\ &= \left(\frac{\partial f}{\partial x}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial r}{\partial y}\right)\frac{\partial}{\partial r} + \left(\frac{\partial f}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \theta}{\partial y}\right)\frac{\partial}{\partial \theta} \end{aligned}$$

Contrast this with the ”naive” attempt at putting the gradient in polar:

$$\frac{\partial f}{\partial r}\frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}$$

which after applying the chain rule yields a different expression. In physics and engineering, we use the orthonormal basis $\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}$; everything is off by a factor of r.

If $\langle , \rangle$ is an inner product on each tangent space, then $\operatorname{grad} f$ is defined by $df_p(v) = \langle (\operatorname{grad} f)_p, v\rangle$ (for $v \in T_pM$). This is called “raising an index”. We’ll see related things later.

Line Integrals, Heuristically

First case of results in Chapters 16 and 17. A covector field is an operator that’s waiting to be fed a vector (“it’s hungry”). This means it is also something to integrate along a curve $\gamma\colon [a, b] \to M$. Indeed, $\int_\gamma \omega$ should be $\int_a^b \omega(\gamma'(t))\, dt$. Now

$$\begin{aligned} \omega(\gamma'(t)) &= \omega(d\gamma(\frac{\partial}{\partial t})) \\ &= (\gamma^*\omega)(\frac{\partial}{\partial t}). \end{aligned}$$

In other words, $\gamma^*\omega = \omega(\gamma'(t))\, dt$ (because they act the same way and we’re in a 1d vector space). Hence

$$\int_\gamma \omega = \int_a^b \gamma^*\omega;$$

we will study these sorts of integrals, and they will give us global topological data (like things about connectedness). This is interesting, because derivatives are local things. We need the curve to be piecewise smooth; so we want to know that for smooth manifolds, connected is equivalent to being connected by piecewise smooth curves (Lemma 11.33).

Line Integrals, Officially

On $\mathbb{R}$, consider a covector field $\omega = f(t)\ dt$. Define $\int_{[a,b]} \omega = \int_a^b f(t)\ dt$.

Proposition 2.1 (11.31)**.** *This quantity is invariant under orientation preserving diffeomorphism.*

Proof. Substitution, or change of variables. □

Exercise 11.32 talks about change of orientation. Now we make the definition

$$\begin{aligned}\int_\gamma \omega &= \int_{[a,b]} \gamma^* \omega \\ &= \int_a^b \omega(\gamma'(t))\ dt\end{aligned}$$

Now extend to piecewise smooth by doing it along each segment and adding up the results.

Example 2.2. The integral is reparametrization independent, but not path independent. Take $\omega = x\ dy$ on $\mathbb{R}^2$, and consider two paths from $(0,0)$ to $(2,2)$; $\gamma_1(t) = (2t, 2t)$ and $\gamma_2(t)$ is piecewise: $(0,t)$ then $(t,2)$ (there is something to say about parametrization independence and not having to match up the parameters). Thus $\gamma_1^*\omega = (2t)2\ dt = 4t\ dt$ (the rule is just plug in). Hence

$$\int_{\gamma_1} \omega = \int_0^1 4t\ dt = 2.$$

Similarly,

$$\int_{\gamma_2} \omega = \int_0^2 0\ dt + \int_0^2 t d(2) = 0.$$

This is the failure of conservation.

Conservative covector fields

Terminology ($\star$ for emphasis):

- **exact** ($\star$) covector field is $\omega = df$, for $f \in C^\infty(M)$.
- f is called a **potential** for ω

  (engineers and physicists use $E = -\operatorname{grad} \phi$ for potential of say, the electric field)

- **conservative** covector fields have $\int_\gamma \omega$ path-independent, i.e. depends only on the endpoints (same as saying integral over closed curves vanishes: this is an early indication why piecewise-smooth is nicer).

  In physics, a vector field X is conservative if $\int_\gamma X \cdot \gamma'(t)\ dt$ is path independent. Here, you're taking $X \cdot \_$ as your covector field.

- A covector field ω is **closed** $(\star)$ if $d\omega = 0$.

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Lecture 3: April 4

Definition 3.1. **exact** means $\omega = df$

conservative means $\int_p^q \omega$ is path-independent

closed means

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j} \qquad (11.21)$$

in every coordinate chart.

Proposition 3.2 (11.45). *(a) closed if and only if (b) at every point, there is a chart in which (11.21) holds if and only if for all* $X, Y \in C^\infty(U)$,

$$X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) = 0 \qquad (11.22).$$

Proof. (a)⇒ (b) is immediate; (b)⇒ (c) is a calculation, not enlightening

(c)⇒ (a): Take $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}$. You plug into (11.22) and get the result. □

The LHS of (11.22) is the invariant definition of $d\omega$:

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

Covector fields are 1-forms, and saying $d\omega = 0$ is equivalent to being closed. In order to be a tensor, you want to only depend on a point. Related to theorem about map between sections being a bundle; same type of thing as detecting a tensor. In other words, we need to show that $d\omega$ is $C^\infty(M)$-linear in all of its slots.

Proposition 3.3 (11.44). *exact ⇒ closed*

Proof. Clairaut's Theorem (equality of mixed partials) yields the result directly. □

A brief preview of De Rham cohomology...

Example 3.4 (11.47). Archetypical example of closed but not exact

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2} = d\theta$$

This works out on $\mathbb{R}^2 \setminus \{(0,0)\}$ minus some ray out of the origin.

Theorem 3.5 (11.42). *Conservative if and only if exact*

Proof. $\Rightarrow$ Fundamental Theorem of line integrals (11.39) says that

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)), \qquad \gamma : [a,b] \to M$$

$\Leftarrow$ Pick a point $p \in M$ (well, one for each component of M). Define $f(q) = \int_p^q \omega$. Conservative ω means that f is well-defined. **Idea:** Show that $df_q = \omega_q$. So pick a coordinate at q. For $\frac{\partial f}{\partial x^i}$, pick a path from p to q that approaches q along an x^i coordinate curve. On the last segment, $\gamma(t) = (0, \cdots, t, 0, \cdots, 0)$ with t in the i component. Now $\gamma'(t) = (0, \cdots, 1, \cdots, 0)$. Hence

$$\begin{aligned} f(\gamma(t)) - f(\gamma(c)) &= \int_c^t \omega(\gamma'(s))\, ds \\ &= \int_c^t \omega^i(\gamma(s))\, ds \\ \Rightarrow \frac{\partial f}{\partial x^i} &= \omega^i(\gamma(t)). \end{aligned}$$

□

Corollary 3.6 (Lee, 11.46)**.** *"closed" and "exact" are invariant under local diffeomorphism*

Definition 3.7. A region of $\mathbb{R}^n$ is **star-shaped** if it is convex with respect to a point.

Proposition 3.8 (Poincaré lemma for covector fields: 11.49)**.** *Let $U \subset \mathbb{R}^n$ be open and star-shaped. Then every closed covector field on U is exact.*

Proof. Define f by integrating ω from the point p with respect to which U is convex. without loss of generality, p is the origin. For $x \in U$, let $\gamma(t) = tx$. Define

$$\begin{aligned} f(x) &= \int_0^1 \omega_{\gamma(t)}(\gamma'(t))\, dt \\ &= \int_0^1 \sum_{i=1}^n \omega_i(tx) x^i\, dt \\ \Rightarrow \frac{\partial f}{\partial x^j} &= \int_0^1 \left[\sum_{i=1}^n \frac{\partial \omega_i}{\partial x^j} t x^i + \omega_j(tx) \right] dt \\ \text{(using closedness)} \quad &= \int_0^1 \left[t \sum_{i=1}^n \frac{\partial \omega_j}{\partial x^i} x^i + \omega_j(tx) \right] dt \qquad (\star) \end{aligned}$$

Recognize that $\sum_{i=1}^n \frac{\partial \omega_j}{\partial x^i} x^i = \frac{d}{dt}(\omega_j(tx))$. Thus

$$\begin{aligned} (\star) &= \int_0^1 t \frac{d}{dt}(\omega_j(tx)) + \omega_j(tx)\, dt \\ &= \int_0^1 \frac{d}{dt}\left(t\omega_j(tx)\right)\, dt \\ &= t\omega_j(tx)\Big|_{t=0}^{t=1} \\ &= \omega_j(x) \end{aligned}$$

□

Example 3.9 (Lee, 11.51)**.** On the star-shaped region $\mathbb{R}^3$, define

$$\omega = e^{y^2}\, dx + 2xye^{y^2}\, dy - 2z\, dz.$$

It is easy to check that ω is closed. We want to find an f for which $\omega = df$. Integrating in x, we get $f(x,y,z) = xe^{y^2} + C_1(y,z)$. Integrating in y, we find $C_1(y,z) = C_1(z)$. Integrating in z yields $C_1(z) = -z^2 + C$. Thus we obtain $f(x,y,z) = xe^{y^2} - z^2$.

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Lecture 4: April 7

Tensors

Covariant k-tensor is a multilinear map $T: V \times \cdots \times V \to \mathbb{R}$; k copies, called the rank of the tensor. The space is called $T^k(V^*) = L(V, \cdots, V; \mathbb{R})$. For contravariant, put $*$; the shorthand is $T^k(V) = L(V^*, \cdots, V^*; \mathbb{R})$. Unify covariant with contravariant by writing

$$T^{(k,l)}(V) = L(V^*, \cdots, V^*, V, \cdots, V; \mathbb{R}) = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$$

(k indices in the first $\cdots$, and l in the second).

Example 4.1. 1. $V = \mathbb{R}^2$ and $T(X, Y) = \det[X\ Y] = X^1Y^2 - X^2Y^1$. Then $T \in L(\mathbb{R}^2, \mathbb{R}^2; \mathbb{R})$. It follows that $T = \varepsilon^1 \otimes \varepsilon^2 - \varepsilon^2 \otimes \varepsilon^1$.

2. Mixed tensor: take $V = \mathbb{R}^3$, and $T(X, Y, \omega) = \omega(X \times Y)$. Express in terms of basis (e_i) for $\mathbb{R}^3$ and dual basis (ε^i)

$$T = \varepsilon^1 \otimes \varepsilon^2 \otimes e_3 - \varepsilon^2 \otimes \varepsilon^1 \otimes e_3 + \cdots$$

3. Curvature tensor: $(3, 1)$ tensor has a bunch of symmetries, in dimension 2 it collapses to a number. There are uses for mixed tensors, but not yet; see the geometric structures course.

4. Inner product. Standard one is $g(X, Y) = X \cdot Y$. A variant: Take $\tilde{g}(X, Y) = X^T AY$ for A positive definite symmetric and nonsingular. (Semireimannian geometry if only semidefinite)

5. Take V finite dimensional and consider $\mathrm{Hom}(V, V)$.

   Define a map $\mathrm{Hom}(V, V) \to L(V^*, V; \mathbb{R})$ by $A \mapsto \tilde{A}: (\omega, v) \to \omega(A(v))$. Then $A \mapsto \tilde{A}$ is injective (check it concretely), and an isomorphism (just count dimensions). By a theorem, $L(V^*, V; \mathbb{R}) \simeq V \otimes V^* \simeq \mathrm{Hom}(V, V)$. To relate them directly, define $\Phi: V \times V^* \to \mathrm{Hom}(V, V)$ by $\Phi(\chi, \eta)(v) = \eta(v)\chi$. Then pass to the quotient (i.e., use characteristic property of abstract tensor product).

   It turns out that Φ is a really bad map; not injective (see $(\chi, 0)$, $(0, \eta)$, or even $(\chi/r, r\eta)$). It's not surjective, because nonsingular homomorphisms are not in the image. There is also a dimension problem between $V \times V^*$ and $\mathrm{Hom}(V, V)$. The tensor product construction fixes all of this, so that $\tilde{\Phi}$ becomes very well-behaved.

   First enlarge $V \times V^*$ to be a free vector space on $V \times V^*$. This takes care of the lack of injectivity of Φ. Then use $\Pi: \mathcal{F}(V \times V^*) \to V \otimes V^*$ to take care of injectivity. Notice the distinction between π and Π; the former is for $V \times V^* \to V \otimes V^*$ whereas the latter is for quotienting into the equivalence classes ensuring bilinearity.

6. 0-tensors. $T^0(V) = T^{(0,0)}(V) = \mathbb{R}$, so 0-tensors are scalars.

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Lecture 5: April 9

Definition 5.1. An element in a tensor product space is **decomposable** if it is a product of elements in the factor spaces.

This will come up when we meet the wedge product (Chapter 14).

Proposition 5.2 (12.7, Characteristic Property of Tensor Products)**.** *Multilinear maps factor through the tensor product. In other words:*

Let $\pi\colon V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_k$ *be the projection. Given a multilinear map* $A\colon V_1 \times \cdots \times V_k \to X$*, there is a unique linear map* $\tilde{A}$ *such that* $A = \tilde{A} \circ \pi$*.*

The π is hiding something. It need not be injective nor surjective; its image consists of decomposables. Start with $V_1 \times V_2 \hookrightarrow F(V_1 \times V_2)$. This is a huge free vector space on all elements (v_1, v_2). Then we compose with Π, which mods out by the linearity relations.

Start with $A\colon V_1 \times V_2 \to X$. Consider $\iota\colon V_1 \times V_2 \to F(V_1 \times V_2)$; by the characterstic property of the free vector space, there is a linear $\overline{A}\colon F(V_1 \times V_2) \to X$. The linearity of $\overline{A}$ is too weak; we need the multilinearity of A. This is "the only way to work with the individuals slots". For instance,

$$\begin{aligned}\overline{A}(v_1 + w_1, v_2) &= A(v_1 + w, v_2)\\ &= A(v_1, v_2) + A(w_1, v_2)\\ &= \overline{A}(v_1, v_2) + \overline{A}(w_1, v_2)\end{aligned}$$

and likewise for multiplication by scalars. Thus $\overline{A}$ respects the relations imposed by Π, so we may pass to the quotient to obtain $\tilde{A}\colon V_1 \otimes V_2 \to X$ such that $\overline{A} = \pi \circ \tilde{A}$.

This property characterizes the tensor product.

Proposition 5.3 (Lee, 12.10)**.**

$$V_1 * \otimes \cdots \otimes V_k^* \simeq L(V_1, \cdots, V_k; \mathbb{R})$$

Proof. Make the diagram $\Phi\colon V_1^* \times \cdots \times V_k^* \to L(V_1, \cdots, V_k; \mathbb{R})$ and $\pi\colon V_1^* \otimes \cdots \otimes V_k^*$. Then by 12.7, there is a unique $\tilde{\Phi}$ such that $\Phi = \tilde{\Phi} \circ \pi$. The map is $\Phi(\omega^1, \cdots, \omega^k)(v_1, \cdots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k)$. So $\tilde{\Phi}$ is unique, but now we must show it is an isomorphism. To do so, we pick a basis and write it down.

$$\Phi(b) = b(E_{(1)\ i_1}, \cdots, E_{(k)\ i_k}) \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k},$$

and now show it is a two-sided inverse.

The alternative is to show Φ is injective, and then do a dimension count argument. □

Symmetric Tensors

Reasons we talk about symmetric tensors:

- In Chapter 13, we meet Riemannian metrics.
- Symmetrization operator; this is a warmup for the antisymmetrization operator, which occurs in Chapter 14 with the wedge product.

Definition 5.4. An operator T is called an **idempotent** or a **projection** when $T^2 = T$.

Consider a projection from $T^k(V^*) \to \Sigma^k(V^*)$. Then the codomain is contained in the domain, and the map is the identity when restricted to its codomain. For $A \in T^k(V^*)$,

$$\operatorname{Sym} A(v_1, \cdots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} A(v_{\sigma(1)}, \cdots, v_{\sigma(k)})$$

where each of the latter terms are written as ${}^{\sigma}A(v_1, \cdots, v_k)$. The reason for the $\frac{1}{k!}$ is to make it the identity on $\Sigma^k(V^*)$.

Definition 5.5. For $S, T \in \Sigma^k(V^*)$, define the **symmetic product**

$$ST = \operatorname{Sym}(S \otimes T).$$

Tensors on Manifolds

First we think in terms of coordinates. Define the k-covariant tensors on M to be $T^k(T^*M) = T^{(0,k)}(TM)$. To show this is a bundle, we're going to need local trivializations. Restrict to a coordinate domain $((x^i), U)$. Then a frame for $T^k(T^*M)|_U = T^k(T^*U)$ is given by $\left(\frac{\partial}{\partial x^i}\right)$ (the frame for $TM|_U$) and (dx^i) the coframe.

Now we have

$$dx^{i_1} \otimes \cdots \otimes dx^{i_k}$$

gives an identification of $T^k(T^*U) \simeq U \times T^k(\mathbb{R}^n)$ (by thinking in coordinates, we've identified U with an open subset of $\mathbb{R}^n$).

Sections

The general symbol is Γ; so the smooth sections are $\Gamma(T^kT^*M)$, or $\mathcal{T}^k(M)$ for short. Taking $k = 1$ is the case of smooth covector fields; proving all the equivalent characterizations of smoothness was a problem from our final last quarter. The generalization to larger k is just the same thing, but with more slots to keep track of.

Next time we'll move on to the vector bundle homomorphism lemma in a larger generalization. When $k = 1$ it the exact same theorem.

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Lecture 6: April 11

Lemma 6.1 (Tensor Characterization Lemma, 12.24)**.**

$$\mathcal{A}: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$$

is $C^\infty(M)$*-multilinear if and only if* $\mathcal{A}$ *arises from a smooth covariant tensor.*

For $k = 1$, this is exactly the bundle homomorphism characterization lemma (10.29). Over there, we had $\mathcal{A}: \mathfrak{X}(M) \to C^\infty(M)$ (thought of as $\Gamma(TM)$ and $\Gamma(M \times \mathbb{R})$ respectively), and that $C^\infty(M)$-linear if and only if there is a smooth covector field A such that $\mathcal{A}(X)_p = A_p(X_p)$.

Proof. The proof follows 10.29, but generalizes the argument to more slots. Recall that 10.29 was the "deja vu" proof, that used bump functions twice. So we will end up using bump functions twice, once for each slot.

1. $\mathcal{A}(\cdots, X_i, \cdots)|_p$ depends only on X_i on a neighborhood of p; so localize.
2. Now pick a frame on some neighborhood. Write $X_i = f^j E_j$ (where the E_j are our frame). We get $\mathcal{A}(\cdots, f^j E_j, \cdots)$. Formally, it is easiest to take two different vector fields that agree at p. Then their f^j's at p will be the same.
3. Repeat for all slots. Define $A_p(v_1, \cdots, v_k) = \mathcal{A}(X_1, \cdots, X_k)$ where $X_i(p) = v_i$. By the preceeding, this is well-defined (independent of our extension). This gives us a rough field, and it is smooth because $A($ smooth fields $) = \mathcal{A}($ smooth fields $)$ which is smooth since $\mathcal{A}$ is given to be C^∞.

We have to jump through these hoops because $\mathcal{A}$ needs its fields to be global. □

Pullbacks

Why is $F^*(A \otimes B) = (F^*A) \otimes (F^*B)$? By definition,

$$\begin{aligned} F^*(A \otimes B)(v, w) &= A \otimes B(dFv, dFw) \\ &= A(dF(v))B(dF(w)) \\ &= (F^*A)(v)F^*B(w) \\ &= (F^*A \otimes F^*B)(v, w) \end{aligned}$$

Note: $F^*(fA)$ is a special case, thinking of f as a 0-tensor. There are secret "composed with f's" all over the place. Thus given $F: M \to N$, we have

$$F^*(fA) = (F^*f)F^*A = (f \circ F)(F^*A).$$

Note $(F^*A)_p = F^*(A_{F(p)}) =$ "$F^*A \circ F$", although no one ever calls it that.

Computationally, the "$\circ F$" in F^* means "plug it in".

Example 6.2. 1. Lee, 12.29. $A = x^2 dy \otimes dy$, $F(r,\theta) = (r\cos\theta, r\sin\theta)$. Then

$$F^*A = r^{-2}\tan^2\theta\ dr \otimes dr + r^{-1}\tan\theta(d\theta \otimes dr + dr \otimes d\theta) + d\theta \otimes d\theta$$

2. $F\colon (r,\theta) \mapsto (r\cos\theta, r\sin\theta, r)$ for $r > 0$. Its image is a cone. Then

$$\begin{aligned} F^*(y\ dx \otimes dz) &= r\sin\theta\ d(r\cos\theta) \otimes dr \\ &= r\sin\theta\cos\theta\ dr \otimes dr - r^2\sin^2\theta\ d\theta \otimes dr \\ F^*(dx \otimes dx) &= (\cos\theta\ dr - r\sin\theta\ d\theta)^2 \\ &= \cos^2\theta\ (dr)^2 - r\cos\theta\sin\theta(dr \otimes d\theta + d\theta \otimes dr) + r^2\sin^2\theta\ (d\theta)^2 \end{aligned}$$

where concatenation/squares refer to symmetric products; we're not doing naive pre-calculus squaring over here!

3. A substantial computation: spherical coordinates.

Standard inner product on $\mathbb{R}^n$; $\bar{g} = \delta_{ij} dx^i \otimes dx^j$. Let F change to spherical coordinates, given by

$$F(\rho,\theta,\phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi) = (x,y,z)$$

Then $F^*\bar{g}$ gives the inner product in spherical coordinates.

$$\begin{aligned} F^*\bar{g} &= (\delta_{ij} \circ F) d(x \circ F) \otimes d(x^i \circ F) = \sum d(x^i \circ f)^2 \\ &= (\sin\phi\cos\theta d\rho + \rho\cos\theta\cos\phi d\phi - \rho\sin\phi\sin\theta\ d\theta)^2 + \cdots \\ &= \sin^2\phi\cos^2\theta\ (d\rho)^2 + \rho^2\cos^2\phi\cos^2\theta\ (d\phi)^2 + \rho^2\sin^2\phi\sin^2\theta\ (d\theta)^2 \\ &\quad + \rho\sin\phi\cos\phi\cos^2\theta(d\rho \otimes d\phi + d\phi \otimes d\rho) \\ &\quad - \rho^2\sin^2\phi\cos\theta\sin\theta(d\rho \otimes d\theta + d\theta \otimes d\rho) \\ &\quad - \rho^2\cos\phi\sin\phi\cos\theta\sin\theta(d\phi \otimes d\theta + d\theta \otimes d\phi) \\ &\quad - \rho^2\cos\phi\sin\phi\cos\theta\sin\theta(d\phi \otimes d\theta + d\theta \otimes d\phi) \\ &\quad + \qquad \cdots\text{(2 more sets of these terms)} \\ &= (d\rho)^2 + \rho^2\ (d\phi)^2 + \rho^2\sin^2\theta\ (d\theta)^2 \end{aligned}$$

Lie derivatives of tensors

The definition is

$$\mathcal{L}_V \mathcal{T}|_p = \left.\frac{d}{dt}\right|_{t=0} \theta_t^*(\mathcal{T}_{\theta_t(p)})$$

If $\mathcal{T}$ is contravariant (for example, a vector field) then switch to using pushforward of the inverse. Recall that $\theta_t^* = d\theta_{-t}$, and θ_t is the flow of V. For each t, it is a diffeomorphism.

Proposition 6.3 (Properties of $\mathcal{L}_V\mathcal{T}$, 12.32)**.**

1. $\mathcal{L}_V f = Vf = df(V)$
2. $\mathcal{L}_V(A \otimes B) = (\mathcal{L}_V A) \otimes B + A \otimes (\mathcal{L}_V B)$
3. $\mathcal{L}_V(\mathcal{T}(X, Y)) = (\mathcal{L}_V\mathcal{T})(X, Y) + \mathcal{T}(\mathcal{L}_V(X), Y) + \mathcal{T}(X, \mathcal{L}_V(Y))$.

These are all product rules, when interpreted in the right way. To prove them, you *could* write everything out in coordinates and use elementary calculus. Or, you *could* go back to the flow definition of the Lie derivative and mimic the "incremental change" proof of the Liebniz rule. But here's another approach:

Recommended. Use flow coordinates. If $V_p \neq 0$, then $V = \frac{\partial}{\partial x^i}$, and you flow is $(t, c_2, \cdots, c_n)$.

$$\mathcal{L}_V\mathcal{T}(Y, Z)|_p = \frac{\partial}{\partial t}\bigg|_{t=0} \mathcal{T}_{(t,c_2,\cdots,c_n)}(Y(t, c_2, \cdots, c_n), Z(t, c_2, \cdots, c_n))$$

Now the result holds on $\operatorname{supp} V$ by a continuity argument. But $M \setminus \operatorname{supp} V$ is an open set on which $\mathcal{L}_V\mathcal{T} = 0$, since $V = 0$. ☐

Math 546 Manifolds Notes by Avi Levy

Lecture 7: April 14

$\mathcal{L}_V \mathcal{T} = 0$ if and only if $\mathcal{T}$ is invariant under the flow of V.

Definition 7.1. A vector field V is called a **Killing field** of the Riemannian metric g when $\mathcal{L}_V g = 0$.

Equivalently, V is a flow by isometries. We feel like there should always be Killing fields, but it turns out they are pretty rare. Generically, a Riemannian metric has no Killing fields.

A weaker condition is that the flow should be volume-preserving. On $\mathbb{R}^2$, consider $\omega = dx \otimes dy - dy \otimes dx$. Then $\omega(X, Y)$ is the area of a parallelogram. Take $V = y\frac{\partial}{\partial x}$, which has fixed points on the x-axis. Then $\mathcal{L}_V \omega = 0$. Geometrically, the field is a horizontal shear. Thus each horizontal layer is shifted, but has the same length. Computationally, use Proposition 12.33. Then

$$\begin{aligned}(\mathcal{L}_V \omega)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= V\left(\omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right) - \omega\left(\mathcal{L}_V \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) - \omega\left(\frac{\partial}{\partial x}, \mathcal{L}_V \frac{\partial}{\partial y}\right) \\ &= 0 - 0 - 0.\end{aligned}$$

In continuum mechanics, $\frac{1}{2}\mathcal{L}_V g$ is the strain associated to the "infinitesimal deformation V".

Riemannian metrics

On each T_pM, the inner product allows you to get an orthonormal frame. On a neighborhood, given any smooth frame we can apply Gram-Schmidt to get a smooth orthonormal frame (use the fact that the formulas to normalize are smooth - see Problem 13.6 and Corollary 13.8).

Warning 1. Start with a coordinate frame and apply Gram-Schmidt. The result may not be a coordinate frame anymore!

For example, consider the unit sphere with coordinate frame

$$\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right).$$

Then $g = \sin^2 \phi (d\theta)^2 + (d\phi)^2$. The frame is already orthogonal, so we just have to normalize. This gives

$$\left(\frac{1}{\sin \phi}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right)$$

Note that this blows up at the north/south poles, as expected.

We claim that there is no coordinate chart for which these are coordinate vectors. Indeed, if $[X, Y] \neq 0$ then X, Y can't be coordinate vectors in the same chart (and recall this condition is if and only if). But it's clear this is the case, whereupon the claim follows.

One could also say that the sphere is not flat, which also yields the claim. But this would bring us into a discussion of curvature, which is postponed until Math 547.

Pulling Back Metrics

Examples with polar and spherical coordinates. Last wekk, we looked at things that were orthogonal. This is not the generic behavior! Note Example 13.17, induced metric on a graph. Recall graph coordinates:

$$X(u^1, \cdots, u^n) = (u^1, \cdots, u^n, f(u)) \qquad (u^i) \text{ does "double-duty" on domain and codomain}$$

Then

$$\begin{aligned} X^* \bar{g} &= (du^1)^2 + \cdots + (du^n)^2 + (df)^2 \\ &= \text{things with lots of cross terms } du^i \otimes du^j \end{aligned}$$

Geometrically, an orthogonal grid will push to a bunch of parallelograms.

Riemannian distance function

The Riemannian metric let's us do geometry! Since g gives rise to a length on vectors, we can get an arclength of a curve $\gamma\colon [a, b] \to M$ by defining

$$L_g(\gamma) = \int_a^b |\gamma'(t)| \, dt.$$

Definition 7.2.

$$d_g(p, q) = \inf_{\substack{\gamma \text{ p.w. smooth} \\ \gamma(a)=p, \gamma(b)=q}} L_g(\gamma)$$

Claim 7.3. d_g *is a metric in the topological sense, and it induces the manifold topology*

Proof. Clearly $d_g(p, q) \geq 0$, $d_g(p, q) = d_g(q, p)$, and $d_g(p, p) = 0$. The triangle inequality comes from the infimum, so it remains to show that $d_g(p, q) > 0$ if $p \neq q$, and the equivalence of the topologies.

We need a linear algebra fact to proceed.

Lemma 7.4 (B.49, B.50, page 638)**.** *All norms on a finite dimensional vector space are equivalent.*

In other words, there are constants c, C such that $cB_{\bar{g}} \leq B_g \leq CB_{\bar{g}}$.

Next, we use local compactness and regular coordinate balls to extend this from vector spaces onto the whole tangent space. Use compactness to get minimum/maximum values for c, C. □

Math 546 Manifolds

Notes by Avi Levy

Lecture 8: April 16

Riemannian distance function, continued

We were in the middle of proving $d_g(p, q) > 0$ if $p \neq q$; this is the last piece of proving that d_g is a (topological) metric. Then, we will show that the topology induced by d_g agrees with the manifold topology. Note that the manifold M must be connected for the distance to be defined; there are various ways to deal with various components.

Lemma 8.1 (Modified version of 13.28)**.** *Given a coordinate domain $U \subset M$, compact $K \subset U$, g a Riemannian metric on M, and $\bar{g}$ be a Riemannian metric on U such that the coordinate fields are orthonormal. Then there are positive constants c, C such that for every $x \in K$ and every $v \in T_xM = T_xU$, we have*

$$c|v|_{\bar{g}} \leq |v|_g \leq C|v|_{\bar{g}}.$$

This is an extension of the linear algebra result mentioned last time, which says that all norms on a finite dimensional vector space are equivalent. We use compactness to extend from a point.

Idea sketch. Pick c, C to be min and max of norm on unit tangent bundle restricted to K; then just verify the inequality. □

Proof of Theorem 13.29, hard parts. Fix $p \in M$, pick a regular coordinate ball V centered at p with radius ϵ in coordinates (distance coming from $\bar{g}$). Note that $\epsilon > 0$ is implied since we have a regular coordinate ball.

For $q \in \partial V$, observe that

$$\begin{aligned} d_g(p, q) &\geq \inf(L_g\gamma) \\ &\geq c \inf(L_{\bar{g}}\gamma) \\ &= c\epsilon \qquad (\star) \end{aligned}$$

The infimum is taken over all piecewise smooth paths in V from p to a point on ∂V. Thus if $p \neq q$, $d_g(p, q) > 0$.

To show the topologies are equivalent, use regular coordinate balls as a basis for the original manifold topology. Use "d_g balls" for the metric topology. By $(\star)$, it follows that when $q \in M \setminus V$, $d_g(p, q) \geq c\epsilon$. Thus if we choose $r < c\epsilon$, the "d_g ball of radius r" centered at p must lie in V.

Conversely, suppose we are given r. Choose $\epsilon < \frac{r}{C}$. Let V be the regular coordinate ball of radius ϵ centered at p. Now $d_g(p, q) = \inf_{p \to q}(L_g\gamma)$. If $\text{Im}(\gamma) \subset V$, then $L_g(\gamma) \leq CL_{\bar{g}}(\gamma) \leq C\epsilon$ for $q \in V$. Now take infimum over these γ; it follows that $d_g(p, q) \leq \inf_{\text{Im}(\gamma) \subset V} \leq C\epsilon$. □

Theorem 8.2. *Every manifold admits a Riemannian metric.*

Proof. By 13.29, each component admits a d_g. Then there is a trick; pick a point in each component and declare them to be one unit apart. □

Corollary 8.3. *Every smooth manifold is metrizable.*

Of course, we already knew that every topological manifold is metrizable (this depends on the Urysohn Metrization Lemma, which we technically didn't cover). However, the metric coming from Urysohn is a lot less relevant than the metric we just put on our space that's coming from Riemannian.

We can talk about completeness now. You can show this is equivalent to saying that bounded sets have compact closure. This is all standard topology.

Comment from Math 547 (because people are interested in curvature): Define geodesics (the analog of "straight lines"). Zero acceleration (this becomes a second order differential equation in local coordinates). We solve first order ODEs by thinking of them as vector fields. So we can solve a second order ODE by using twice as many variables: it is a first order ODE on TM. The vector field is called the "geodesic spray".

The manifold doesn't have a curve going through it, it has a curve going in each direction: all the curves spraying out from the point in all directions.

We defined completeness for vector fields. So now we can talk about curves up in TM; you project them down and you get the geodesics. Now completeness of the geodesic spray means that the integral curves are defined for all time.

Definition 8.4. Geodesic completeness means that the geodesic spray is complete; all the geodesics are defined on all of $\mathbb{R}$.

A big theorem in 547 is that geodesic completeness is equivalent to metric space completeness.

Math 546 Manifolds Notes by Avi Levy

Lecture 9: April 18

The musical isomorphisms

In general, we can construct a non-canonical isomorphism from $T_pM \to T_p^*M$. Given an inner product, there **is** a canonical isomorphism. Given an inner product g, identify a vector X with a covector ω defined by $\omega(Y) = g(X, Y)$; i.e., $\omega = g(X, \cdot)$. In terms of indices,

$$\begin{aligned} g(X,\cdot) &= \left(g_{ij}\,dx^i \otimes dx^j\right)\left(X^k \frac{\partial}{\partial x^k}\right) \\ &= g_{ij}\left(dx^i\left\{\frac{\partial}{\partial x^k}\right\}\right) X^k\,dx^j \\ &= g_{ij}\delta^i_k X^k\,dx^j \\ &= g_{ij}X^i\,dx^j. \end{aligned}$$

If you're a true index-pusher, you would write $\omega = \omega_j\,dx^j$ and $\omega_j = g_{ij}X^i$. We write $\omega = X^\flat$; we **define** $X_i = g_{ij}X^j$.

Definition 9.1 (Preview of page 358:)**.** We define **interior multiplication of** X **into** g to be

$$i_X g = g(X, \cdot).$$

If we choose coordinates so that $\frac{\partial}{\partial x^i}$ is orthonormal at a point p, then $X_i(p) = X^i(p)$. At p, the isomorphism $\flat\colon T_pM \to T_p^*M$ just takes the coordinate basis to the dual basis:

$$\left(\frac{\partial}{\partial x^i}\right)^\flat = dx^i.$$

Let $\#\colon T_p^M \to T_pM$ be the inverse of $\flat$. For $\omega = \omega_i\,dx^i$, define $\omega^\#$ by setting $g(\omega^\#, \cdot) = \omega$. Then

$$\omega^\# = \omega^j \frac{\partial}{\partial x^j}, \qquad g_{ij}\omega^j = \omega \qquad (\star)$$

If (L^{ij}) is the matrix that gives the map $(\omega_j) \to (\omega^i)$, then $L^{ij}\omega_j = \omega^i$. Apply g_{ki} to obtain

$$\begin{aligned} \left(g_{ki}L^{ij}\right)\omega_j &= g_{ki}\omega^i \\ &= \omega_k. \end{aligned}$$

Thus $g_{ki}L^{ij} = \delta^j_k$. Hence $L = g^{-1}$, so we write L^{ij} as g^{ij}.

To illustrate this notation, observe that $L^{ki}g_{ij}L^{jp}$ becomes $L = g^{-1}$ (mixing index notation with matrix notation). Let g^{ij} denote the contravariant 2-tensor

$$S = g^{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

Claim 9.2. *S is an inner product for covectors, and* $S(\omega,\eta) = g(\omega^\#,\eta^\#)$.

Proof.

$$\begin{aligned} S^{ij} &= S(dx^i, dx^j) \\ &= g^{ij}. \end{aligned}$$

On the other hand, $dx^{i\#} = g^{ik}\frac{\partial}{\partial x^k}$. Thus

$$\begin{aligned} g(dx^{i\#}, dx^{j\#}) &= g\left(g^{ik}\frac{\partial}{\partial x^k}, g^{jp}\frac{\partial}{\partial x^p}\right) \\ &= (g_{rs}\,dx^r \otimes dx^s)\left(g^{ik}\frac{\partial}{\partial x^k}, g^{jp}\frac{\partial}{\partial x^p}\right) \\ &= g_{rs}g^{ik}g^{jp}\,dx^r\left(\frac{\partial}{\partial x^k}\right)dx^s\left(\frac{\partial}{\partial x^p}\right) \\ &= g_{rs}g^{ik}g^{jp}\delta^r_k\delta^s_p \\ &= g_{rs}g^{ir}g^{js} \\ &= \delta^i_s g^{js} \\ &= g^{ji} \end{aligned}$$

So somewhere we were sloppy with g being symmetric, but this ends up yielding the result. □

Example 9.3 (Polar coordinates)**.** $g = dr^2 + r^2\,d\theta^2$. Now

$$\left|\frac{\partial}{\partial r}\right|_g = 1, \qquad \left|\frac{\partial}{\partial \theta}\right|_g = r.$$

The corresponding orthonormal frame is

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \end{pmatrix} \to^{\flat} \begin{pmatrix} dr \\ r\,d\theta \end{pmatrix}$$

Likewise,

$$\#: \begin{cases} dr \mapsto \frac{\partial}{\partial r} \\ d\theta \mapsto \frac{1}{r^2}\frac{\partial}{\partial \theta} \end{cases}.$$

Thus $g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix}$. Observe that the operator norm $|d\theta| = \frac{1}{r}$, which matches our intuition.

Example 9.4 (Gradient)**.** In standard Euclidean, $\operatorname{grad} f = (df)^\#$. In general,

$$\begin{aligned} (df)^\# &= g^\#\left(\frac{\partial f}{\partial x^i}dx^i, \cdot\right) \\ &= g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}. \end{aligned}$$

Flatness

Definition 9.5. A Riemannian metric g is **flat** if there is a local isometry to $(\mathbb{R}^n, \bar{g})$.

This occurs if and only if at every $p \in M$, there are local coordinates in which $g = \bar{g}$ (i.e., $g = \phi^* \bar{g}$).

Remark (Compare and contrast). Smooth manifolds: locally diffeomorphic to $\mathbb{R}^n$.

Flat manifolds: locally isometric to $(\mathbb{R}^n, \bar{g})$.

By Gram-Schmidt, we can always find a local orthonormal frame; but it won't necessarily be a coordinate frame.

Example 9.6 (From the 545 final)**.** We showed that $\mathbb{S}^3$ is isomorphic to the unit quaternions, which is isomorphic as a Lie group to $SU(2)$. We have a global frame i, j, k with the ambient Riemannian metric inherited from $\mathbb{R}^4$. But from another problem from last quarter, we know that this is isomorphic as a Lie algebra to $(\mathbb{R}^3, \times)$ (cross product).

Math 546 Manifolds Notes by Avi Levy

Lecture 10: April 21

Index pushing

Fixing the $g^{ij}dx^j$ from last time.

$g^\#$ means something like $g^{\#\#}$. For instance

$$\begin{aligned} g^\#(\eta,\cdot) &= g^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^i}\left(\eta_k\, dx^k,\cdot\right) \\ &= g^{ij}\eta_k\delta_i^k \\ &= g^{ij}\eta_i\frac{\partial}{\partial x^i} \\ &= \eta^j\frac{\partial}{\partial x^j}. \end{aligned}$$

If you're a pro, you would write $g^{ij}\eta_i = \eta^j$.

If $\eta = df$, then $\eta_i = f_{,i}$ where , means derivative. You would write $\operatorname{grad} f = f^{,j}\frac{\partial}{\partial x^j}$.

Alternating Tensors (Ch. 14)

They live in the **Grassman** or **Exterior** Algebra. Unifies and extends ideas from vector calculus.

Example 10.1.

1. Cross product: introduced as a vector (i.e., contravariant tensor). However, we care about it because it gives signed area.
2. grad f: input is vector, output is a number.
3. volume element on a surface in $\mathbb{R}^3$: $i_N dV$ where $dV = dx^d y^d z$, N is a unit normal vector.

All these things were introduced as vectors, but they were really covectors all along. We have been implicitly using musical isomorphisms to identify things in elementary calculus.

Elementary Alternating Tensors

Let's say we live in V, a vector space of dimension n.

Definition 10.2. Given $I = (i_1, \cdots, i_k)$, define the **elementary alternating tensor**

$$\varepsilon^I = \begin{cases} 0, & k > n \\ (14.1) & k \le n. \end{cases}$$

As a consequence, $\varepsilon^I(E_{j_1}, \cdots, E_{j_k}) = \delta^I_J$ where $J = (j_1, \cdots, j_k)$.

Definition 10.3. The **generalized Kronecker symbol** is

$$\delta^I_J = \begin{cases} \operatorname{sgn} \sigma, & \sigma\colon I \to J \\ 0, & \text{else} \end{cases}$$

Observe that $\varepsilon^I(E_1, \cdots, E_k) = 1$. Since ε^I is alternating,

$$\varepsilon^I = k! \operatorname{Alt}(\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k})$$

Geometric Interpretation

Think along the lines of covectors as "vector detectors". So ε^2 is a family of planes at unit intervals, perpendicular to x^2 axis. For $2\varepsilon^2$, the planes would be spaced at 1/2-unit intervals. Then $\varepsilon^2(v)$ is the number of planes pierced by v.

For $\varepsilon^1 \wedge \varepsilon^2$, you get a "grid of planes". You feed it a pair of vectors, i.e. a parallelogram. Then $\varepsilon^1 \wedge \varepsilon^2(v_1, v_2)$ is the number of "tubes" pierced by the parallelogram. Really, $\varepsilon^1 \wedge \varepsilon^2(v_1, v_2)$ is the signed area of the parallelogram projected onto the xy-plane.

Wedge Product

The "shuffle" definition:

Definition 10.4. The **wedge product** of $\omega \in \Lambda^k(V^*), \eta \in \Lambda^l(V^*)$ is

$$\omega \wedge \eta(v_1, \cdots, v_{k+l}) = \sum_{\sigma \in S_{(k,l)}} (\operatorname{sgn} \sigma) \omega(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$

where $S_{(k,l)}$ is the set of (k,l) shuffles:

$$\sigma(1) < \cdots < \sigma(k) \text{ and } \sigma(k+1) < \cdots < \sigma(k+l)$$

We claim that this is equivalent to the book's definition. Indeed, each term in the shuffle definition corresponds to $k!l!$ equal "unshuffle" terms, obtained by rearranges of ω arguments and η arguments. Now a little calculation shows that

$$\omega \wedge \eta(v_1, \cdots, v_{k+l}) = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta)(v_1, \cdots, v_{k+l}).$$

Thus we match the book's definition, and this finally shows that the "shuffle definition" yields an alternating tensor.

Lemma 10.5 (Lee, 14.10)**.** $\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}$

Proof. It suffices to apply both sides to a collection of basis vectors $(E_{p_1}, \cdots, E_{p_{k+l}})$. Let $P = (p_1, \cdots, p_{k+l})$ and usual conventions for I, J. The condition on P that makes both sides 0 is: P has a repeated index, any index in P is not in either I or J. If I, J have indices in common, then both sides are 0. Thus we have reduced to the case when the list of all indices appearing in either I or J is distinct.

Since both sides are alternating, without loss of generality $P = IJ$ and I, J are increasing. Now look at

$$\varepsilon^I \wedge \varepsilon^J (E_{i_1}, \cdots, E_{i_k}, E_{j_1}, \cdots, E_{j_k}),$$

but it's trivial to check that these two are the same things, because they're only non-zero in a very specific case. □

We're just checking things over a basis, and the basis is very sparse.

Math 546 Manifolds Notes by Avi Levy

Lecture 11: April 23

Review from last time

For $\omega \in \Lambda^k(V^*), \eta \in \Lambda^l(V^*)$, we have the shuffle definition

$$\omega \wedge \eta(v_1, \cdots, v_{k+l}) = \sum_{\sigma \in S_{(k,l)}} (\operatorname{sgn} \sigma) \omega(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$

Proposition 11.1 (Properties of $\wedge$, 14.11)**.**

(a) bilinear

(b) associative

(c) anticommutative with respect to the usual grading: $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$

(d) $\varepsilon^I = \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$

(e) For covectors $\omega^1, \cdots, \omega^k$,

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \cdots, v_k) = \det(\omega^i(v_j))$$

(f) The shuffle definition extends:

$$\omega \wedge \eta \wedge \phi = \sum_{\sigma \in S_{(k,l,p)}} (\operatorname{sgn} \sigma)\ {}^{\sigma}(\omega \otimes \eta \otimes \phi)$$

for $\phi \in \Lambda^p(V^*)$.

Now we transplant things from a single point (i.e., vector space) onto a manifold. Define the bundle of alternating covariant k-tensors, $\Lambda^k(T^*M)$, which is a smooth subbundle of $T^k(T^*M)$.

Sections

Definition 11.2. We are concerned with **alternating covariant k-tensor fields**, also known as (differential) **k-forms**. The sections are denoted by $\Omega^k(M)$. Familiar cases include $\Omega^0(M) = C^\infty(M)$ and $\Omega^1(M) = \mathcal{T}^1(M) = \Gamma(T^*M)$.

Given any local coframe $(\varepsilon^1, \cdots, \varepsilon^n)$ (i.e., frame for T^*M), we wedge them to get (ε^I), which is a local frame for $\Lambda^k(T^*M)$. For a given k, use ε^I for all $I = (i_1, \cdots, i_k)$ and $i_1 < \cdots < i_k$.

Pullback

This is the same as for general covariant tensors. It distributes over $\wedge$, because it distributes over $\otimes$:

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta).$$

Recall that for $u \in C^\infty(N)$, $F^*u = u \circ F$. Then $F^*du = d(u \circ F) = d(F^*u)$. The pullback for top degree forms is (14.15), so if $F\colon M \to N$ both with dimension n, we obtain

$$F^*(u\, dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) \det(DF)\, dx^1 \wedge \cdots \wedge dx^n.$$

Proof.

$$\begin{aligned} F^*(dy^1 \wedge \cdots \wedge dy^n) &= (F^* dy^1) \wedge \cdots \wedge F^*(dy^n) \\ &= d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F) \\ &= dF^1 \wedge \cdots \wedge dF^n \\ &= G\, dx^1 \wedge \cdots \wedge dx^n, \end{aligned}$$

for some function G (by a dimension argument). We can determine G by evaluating on a basis; i.e.,

$$\begin{aligned} G &= dF^1 \wedge \cdots \wedge dF^n \left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n} \right) \\ &= \det\left(dF^i \left(\frac{\partial}{\partial x^j} \right) \right) \\ &= \det\left(\frac{\partial}{F^i} \partial x^j \right) \\ &= \det(DF), \end{aligned}$$

where D is Lee's name for the Jacobian determinant; this is the differential when expressed in the standard basis for $\mathbb{R}^n$. □

Exterior Derivative

Start out by working on an open subset $U \subset \mathbb{R}^n$. A smooth k-form is then a linear combination of terms like

$$\omega = f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \qquad f \in C^\infty(U).$$

Then define $dw = df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where df is the differential of f; it is a covector field on U, i.e. $df \in T^*U$.

Proposition 11.3 (Properties of d, 14.23)**.** *Consider* $d\colon \Omega^k(U) \to \Omega^{k+1}(U)$.

(a) d *is* $\mathbb{R}$*-linear*

(b) If ω *is a* k*-form, then*

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$$

(c) $d^2 = 0$

(d) $d \circ F^* = F^* \circ d$

Proof. (a) By definition

(b) Let $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Then

$$\begin{aligned} d(u\, dx^I \wedge v\, dx^J) &= d(uv\, dx^I \wedge dx^J) \\ &= d(uv) \wedge dx^I \wedge dx^J \\ &= (v\, du + u\, dv) \wedge dx^I \wedge dx^J \\ &= v\, du \wedge dx^I \wedge dx^J + u\, dv \wedge dx^I \wedge dx^J \\ &= (du \wedge dx^I) \wedge (v\, dx^J) + (-1)^k u\, dx^I \wedge (dv \wedge dx^J) \\ &= d(u\, dx^I) \wedge (v\, dx^j) + (-1)^k u\, dx^i \wedge d(v\, dx^J) \end{aligned}$$

(c) For 0-forms, let's look at $\mathbb{R}^2$ first for simplicity. Then $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$, and

$$\begin{aligned} d^2 f &= d(df) \\ &= \left(\frac{\partial^2 f}{\partial x^2} dx + \frac{\partial^2 f}{\partial x \partial y} dy \right) \wedge dx + \cdots \\ &= \frac{\partial^2 f}{\partial x \partial y} dy \wedge dx + \frac{\partial^2 f}{\partial y \partial x} dx \wedge dy \\ &= \left(-\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y} \right) dx \wedge dy \\ &= 0, \end{aligned}$$

by Clairaut's Theorem. To extend to $\mathbb{R}^n$, it's the exact same idea with more notation:

$$\begin{aligned} d(df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = d^2 f \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ + \cdots + df \wedge dx^{i_1} \wedge \cdots \wedge d^2(x^{i_m}) \wedge \cdots \wedge dx^{i_k} + \cdots \end{aligned}$$

(d) Take 0-forms, and use that F^* distributes over $\wedge$ and $F^*(du) = d(u \circ F)$.

□

Math 546 Manifolds

Notes by Avi Levy

Lecture 12: April 25

Homework discussion: 12-12 modified

Formula (12.10) defines an operator; the main task is to show that the result is a tensor. The orginal problem was as follows:

Define $D_V : \mathcal{T}^k(M) \to \mathcal{T}^k(M)$ by:

(a) $\mathbb{R}$-linear

(b) action on $\mathcal{T}^0 = C^\infty(M)$

(c) distributes over $\otimes$

(d) action on $\mathcal{T}^1$

Given $A \in \mathcal{T}^k(M)$, the fact that $D_v A$ is a tensor does not tell us that $D_V A$ depends locally on A.

Example 12.1. Take $M = \mathbb{S}^1$, with the operator that averages coefficients with respect to $d\theta$ over $\mathbb{S}^1$.

You have to prove from (a-c) that if $A = B$ on a neighborhood U of p then $D_V A|_p = D_V B|_p$.

Proof. Define $C = A - B$; then $C = 0$ on a neighborhood $p \in U$. Take a bump function $\psi = 1$ on a smaller neighborhood $p \in V \subset U$, and $\operatorname{supp} \psi \subset U$. Then $\psi(C) \equiv 0$ as a k-tensor. By (a) then (c),

$$\begin{aligned} 0 &= D_V(\psi(C)) \\ &= (D_V \psi) \otimes C + \psi D_V C. \end{aligned}$$

Evaluating at p,

$$0 = 0 \otimes C|_p + D_V C|_p,$$

and we obtain the desired statement. □

Exterior Derivative, reprise

Last time, we defined the exterior derivative on $\mathbb{R}^n$ by setting

$$d(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \qquad (a).$$

Then we derived the properties:

(b) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$

(c) $d^2 = 0$

(d) $d \circ F^* = F^* \circ d$

Now we extend to manifolds.

Theorem 12.2 (Exterior derivatives on manifolds, 12.24)**.**

Existence Define on coordinates using $\mathbb{R}^n$ formula, show it is well-defined under change of coordinates.

Uniqueness Show that (a-c) (which we now refer to as (i-iii)) and

$$d : C^\infty(M) \to \Omega^1(M) \text{ is differential} \qquad (iv)$$

uniquely defines d.

Proof. For existence, we take overlapping charts (U, ϕ) and (V, ψ). Consider $\omega \in T_p^* M$, for $p \in U \cap V$. Look at $(\Psi^{-1})^* \omega$ (living on $\Psi(U \cap V)$ and $(\Phi^{-1})^* \omega$. The map $\psi^{-1*} \circ \phi^* = (\phi \circ \psi^{-1})^*$ takes the latter to the former.

To show that dw is defined locally, we must verify that

$$\phi^* d((\phi^{-1})^* \omega) = \psi^* d((\psi^{-1})^* \omega).$$

Now property (d) applies to $(\phi \circ \psi^{-1})^*$, since it lives in Euclidean. Hence

$$\begin{aligned} d(\phi^{-1*} \omega) &= d\left((\phi \circ \psi^{-1})^* \phi^{-1*} \omega\right) \\ &= (\phi \circ \psi^{-1})^* d(\phi^{-1*} \omega), \end{aligned}$$

by property (d) on the maps between copies of $\mathbb{R}^n$. Applying ψ^* to both sides, we get the desired result and existence follows.

For uniqueness, we use a "deja vu" argument (double use of bump function). Suppose that $D : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfies (a-c) and (iv), and that we have $\omega_1, \omega_2 \in \omega^k(M)$ with $\omega_1 = \omega_2$ on an open set U; set $\omega = \omega_1 - \omega_2$.

For all $p \in U$, take a bump function ψ such that $\psi = 1$ on a neighborhood $V \ni p$ and $\operatorname{supp} \psi \subset U$. Then $\eta = \psi\omega$ is identically zero on k-forms. By $\mathbb{R}$-linearity, $D\eta = 0$. So

$$\begin{aligned} 0 &= D(\psi\omega) \\ &= (D\psi)\omega + \psi D\omega \qquad (ii) \\ \implies 0 &= 0\omega + D\omega_1 - D\omega_2 \qquad (i) \text{ and } (iv), \end{aligned}$$

where the last step follows by evaluating at p. Now we work locally, so that ω is a linear combination of terms like $f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. But

$$\begin{aligned} D(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}) &= Df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ &\qquad + \cdots \text{alternating terms involving } d^2 \qquad \text{(ii)} \\ &= Df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad \text{(iii)} \\ &= df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad \text{(iv)}, \end{aligned}$$

which matches our local formula. □

Relating d to vector calculus

Use interior multiplication, denoted by i_X (or a funny latex symbol). This is sometimes called "anti-derivation". There is no differentiation going on, but there is a product rule with (-1).

Lemma 12.3 (Lee, 14.13)**.** $(i_X)^2 = 0$ *on forms, and*

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge (i_X \eta).$$

On $\mathfrak{X}(\mathbb{R}^3)$, define

$$\beta \colon \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3), \qquad X \mapsto i_X\, dx \wedge dy \wedge dz.$$

Remark. If X is a unit normal vector field on surfaces, then $i_X\, dx \wedge dy \wedge dz$ is the area form on the surface.

Example 12.4 (Lee, 14.27)**.** Take the field

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z},$$

and observe that

$$\begin{aligned} d(X^\flat) &= d(P\, dx + Q\, dy + R\, dz) \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx \\ &= \beta\left[\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\frac{\partial}{\partial z} + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial}{\partial x} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial}{\partial y}\right] \\ &= \beta(\operatorname{curl} X). \end{aligned}$$

Also define the Hodge star operator,

$$\star \colon C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3), \qquad f \mapsto f\, dx \wedge dy \wedge dz.$$

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Lecture 13: April 28

Last time, we saw that

$$\begin{array}{ccc} \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\flat} & \Omega^1(\mathbb{R}^3) \\ \downarrow \beta & & \downarrow \text{curl} \\ \Omega^2(\mathbb{R}^3) & \xleftarrow{d} & \Omega^1(\mathbb{R}^3) \end{array}$$

We also define $*\colon \Omega^0(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$, $f \mapsto *f = f\, dx \wedge dy \wedge dz$. Writing $X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$, we compute that $d(\beta X)) = * \operatorname{div} X$.

The last two computations yield $\operatorname{curl} \operatorname{grad} X = 0$ and $\operatorname{div} \operatorname{curl} X = 0$, as consequences of $d^2 = 0$.

Moving on to the diagram on page 368, we can generalize to (M^3, g) on page 426. The Hodge start operator $*\colon \Omega^k \to \Omega^{n-k}$ can be made more general; $\beta \circ \#$ is another instance of $*$. See problems 16-18 in Chapter 14; there is also a relation to ∇, see problem 2-3 from Chapter 17.

Invariant Formula for d

See propositions 14.29 and 14.32. These are useful when you want to work in a frame that's not a coordinate frame (i.e., doesn't commute). The main advertisement is Lie groups; the basis for the Lie algebra need not commute.

In general,

$$\begin{aligned} d\omega(X_1, \cdots, X_k) = &\sum_{1 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k) \\ &- \sum_{j=1}^{k} (-1)^j X_j \omega(X_1, \cdots, \hat{X}_j, \cdots, X_k) \end{aligned}$$

and this can be taken as the definition of d.

Proof. Let $D\omega$ be the operator on the right side of the equation. First, show that $D\omega$ is a tensor. Then, evaluate each side at a point to get equality (uses the fact that both sides are tensors).

For $k = 1$, we know that $D\omega$ is alternating. Hence it suffices to consider

$$\begin{aligned} D\omega(fX_1, X_2) &= fX_1(\omega(X_2)) - X_2\omega(fX_1) - \omega([fX_1, X_2]) \\ &= fX_1(\omega(X_2)) - (X_2 f)\omega(X_1) - fX_2(\omega(X_1)) - f\omega([X_1, X_2]) + (Xf_2)\omega(X_1) \\ &= fD\omega(X_1, X_2). \end{aligned}$$

The same manipulation goes through for large k, but you've got to keep track of things.

Now we show that $d\omega_p = D\omega_p$. From before, they are both tensors and linear on ω. Since they are tensors, we can work locally to establish the result. Thus we take $\omega = u\, dx^I$ with I an increasing multi-index, and show that they agree on the fields $\frac{\partial}{\partial x^J}$ for another increasing multi-index J. So we just show that

$$D\omega\left(\frac{\partial}{\partial x^J}\right)\bigg|_p = d\omega\left(\frac{\partial}{\partial x^J}\right)\bigg|_p,$$

and now the result follows. □

Illustration of the book's proof for verifying the exterior derivative of $\omega = u\, dx^{i_1} \wedge dx^{i_2}$:

$$d\omega\left(\frac{\partial}{\partial x^{j_1}}, \frac{\partial}{\partial x^{j_2}}, \frac{\partial}{\partial x^{j_3}}\right) = \sum_{k=1}^{n} \frac{\partial u}{\partial x^k} \delta^{(k,i_1,i_2)}_{(j_1,j_2,j_3)}$$

$$D\omega\left(\frac{\partial}{\partial x^{j_1}}, \frac{\partial}{\partial x^{j_2}}, \frac{\partial}{\partial x^{j_3}}\right) = \sum_{p=1}^{3} (-1)^{p-1} \frac{\partial u}{\partial x^{j_p}} \delta^I_{\widehat{J_p}}.$$

Note that the "correction" term vanishes, since we're working on a coordinate frame. Now $\delta^{kI}_J = 0$ unless $k = j_p$ for some p, and the other two j indices are in I. In these cases,

$$\delta^{kI}_J = (-1)^{p-1} \delta^{kI}_{j_p \widehat{J_p}}.$$

This is non-zero only when $k = j_p$, when it equals $(-1)^{p-1}\delta^I_{\widehat{J}_p}$. Thus the two expressions are the same.

Lie Derivatives of Forms

Theorem 13.1 (Cartan's Formula)**.** $\mathcal{L}_V = i_V d + d i_V$

Proof. For $k = 0$, it is trivial (since $i_V f$ vanishes). Now for the inductive step, write any $k \geq 1$ form as a sum of terms $du \wedge B$. Then use $\mathcal{L}_V(du) = d\mathcal{L}_V u$ (which we proved in Chapter 12). □

Math 546 Manifolds Notes by Avi Levy

Lecture 14: April 30

More complicated foliations

Example 14.1 (Lee, 19.18)**.** 19.18(g) "leaves" are submanifolds that are not all diffeomorphic.

For Lie subgroups $H \subset G$, the cosets of H are a foliation.

Smoothness of a distribution D:

Lemma 14.2 (10.32)**.** *Local frame condition for smoothness of submanifolds.*

Example 14.3 (19.1 d)**.** This is the first example that's not tangent to a foliation. Consider the global frame $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, Y = \frac{\partial}{\partial y}$.

Proof that it's not tangent. To show it's not tangent, start with any ball B around $(0,0,0)$. For each point $(a,b,c) \in B$, there is a piecewise smooth curve with segments that are integral curves of X or Y going from $(0,0,0)$ to (a,b,c). Indeed, it suffices to get to $(0,0,c)$; to do this, make $c = d^2$ for $d > 0$. Then follow Y to $(0,d,0)$, follow X to (d,d,d^2), follow $-Y$ to $(d,0,d^2)$ and then follow $-X$ to $(0,0,d^2)$.

Similarly we can go to all the other points. □

There is some argument here about non-existence; if it were tangent to a foliation, we couldn't fill up the whole $\mathbb{R}^3$ for countability reasons: this is another reason why it's nice to have second-countability (the first and major reason is for partitions of unity).

Ball rolling without slipping: position and orientation of ball, so state space is $\mathbb{R}^2 \times SO(3)$ which is a 5-dimensional space. But the constraint is 2-dimensional.

Frobenius Theorem

Nice properties for a distribution D, tn order of increasing niceness:

(a) Through every p, there's an integral submanifold (D is **integrable**)

(b) If X, Y are sections of D, so is $[X, Y]$ (D is **involutive**)

(c) At every point, there is a flat chart (i.e., where $(\frac{\partial}{\partial x^i})$ is a local frame for D) (D is **completely integrable**)

These are all equivalent! By the Frobenius theorem (filling the crucial step (b) implies (c)). Notice that (c) implies (a) is immediate, and (a) implies (b) by naturality of Lie brackets.

Proof sketch of Frobenius. Work locally; choose a local frame for D, and then modify the elements in the frame so that the elements commute. Then we use the canonical form for commuting vector fields to get our flat local coordinates. If you have a rank-k distribution, then you can pick a set of $n - k$ coordinates so that the $(n - k)$-coordinate plane that corresponds to them is transverse to D. (Use a linear algebra lemma from the appendix)

Reorder so that the $n - k$ coordinates are last, and project onto the first k coordinates. Now just lift the basis coordinates up into D, and show that those will give the desired plane. □

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Lecture 15: May 2

Frobenius Theorem 19.12

$\mathcal{D}$ is involutive implies it is completely integrable.

Proof outline from last time. Since the conclusion is local, work in a local frame (X_i) for $\mathcal{D}$. Pick coordinates so $\mathcal{D}$ projects bijectively onto a coordinate plane $\left(\frac{\partial}{\partial x^i}\right)_{i \le k}$. Then lift the coordinate vector frame to a commuting frame for D. The canonical form for commuting vector fields gives a flat coordinate chart. □

We illustrate these steps with an example.

Example 15.1. Frame for $\mathcal{D}$ on $y > 0$ is

$$X = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}$$
$$Y = y^2\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z}.$$

Then $[X, Y] = -yX \in \mathcal{D}$ (then use a lemma to extend this to all vector fields in $\mathcal{D}$).

Choose $\pi(x, y, z) = (x, y)$. Then

$$d\pi(X) = y\frac{\partial}{\partial x}$$
$$d\pi(Y) = y^2\frac{\partial}{\partial y}.$$

These could be much more complicated, but the point is that since they are linearly independent, you can solve them for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. Once these are solved for, apply $(d\pi|_{\mathcal{D}})^{-1}$ to get a new frame for $\mathcal{D}$.

In our example,

$$V_q = (d\pi|_q)^{-1}\left(\frac{\partial}{\partial x}\right) = \frac{1}{y}X$$
$$W_q = (d\pi|_q)^{-1}\left(\frac{\partial}{\partial y}\right) = \frac{1}{y^2}Y.$$

These fields depend smoothly on q, and they commute (we can just check by hand). In the theorem, we use naturality of brackets to show this; V is π-related to $\frac{\partial}{\partial X}$, and likewise for W; so $d\pi([V, W]) = 0$. Now $d\pi|_{\mathcal{D}}$ is injective. Thus since $[V, W] \in \mathcal{D}$, it follows that $[V, W] = 0$.

In this example,

$$V = \frac{\partial}{\partial x} + \frac{z}{y}\frac{\partial}{\partial z}$$
$$W = \frac{\partial}{\partial y} - \frac{xz}{y^2}\frac{\partial}{\partial z}$$

For commuting coordinates, we need to flow out along a space complementary to $\mathcal{D}$. Since $n = 3$ and $k = 2$, we need to start with a 1-dimensional submanifold. Choosing a fiber of π will suffice; so set

$$S = \{(0,1,c) : c \in \mathbb{R}\}.$$

The flow of V starting at $(0,1,c)$ has parameter s. Then

$$\dot{x} = 1$$
$$\dot{y} = 0$$
$$\dot{z} = \frac{z}{y}$$

which yields the flow

$$(x,y,z) = (s,1,ce^s).$$

Now we flow out along W from this point, using a new parameter t.

$$\dot{x} = 0$$
$$\dot{y} = 1$$
$$\dot{z} = \frac{-xz}{y^2}$$

which is a little more nontrivial; an intermediate computation is $\dot{z} = -\frac{sz}{(1+t)^2}$, whereupon $z = ce^s \exp\left(\frac{s}{1+t} - s\right)$. This gives the integral submanifold

$$(x,y,z) = \left(s, 1+t, ce^{s/(1+t)}\right) \implies (s,t,c) = (x, y-1, ze^{-x/y}).$$

Note that the graph of $z = ce^{x/y}$, is an integral submanifold of $\mathcal{D}$.

There is an application to homogeneous PDE theory:

Suppose $Xf = 0 = Yf$, with initial conditions $f(0,1,z) = g(z)$. To solve this, extend f off S to be constant on integral submanifolds. Then $f(x,y,ce^{x/y}) = g(c)$, so $f(x,y,z) = g(ze^{-x/y})$.

Distributions in terms of flows

We can give $\mathcal{D}$ the local frame $X_1, \cdots, X_k$. In a dual manner, we could specify 1-forms $\omega^1, \cdots, \omega^{n-k}$ such that

$$\mathcal{D} = \bigcap_{j=1}^{n-k} \ker \omega^j.$$

Condition for involutivity in terms of forms? Well, the information in $[X_i, X_j]$ should be encoded in $d\omega^j$.

Definition 15.2. A p-form η **annihilates** $\mathcal{D}$ on some set U if

$$\eta(V_1, \cdots, V_p) = 0, \qquad \forall V_1, \cdots, V_p \subset \mathcal{D}.$$

This is a weaker condition than $i_V\eta = 0$ for all $V \subset \mathcal{D}$.

Definition 15.3. The **annihilator of** $\mathcal{D}$ is

$$I(D) = \{\eta \in \Omega^*(M) : \eta \text{ annihilates } \mathcal{D}\}.$$

Let $\{\omega^i\}_{i=1}^{n-k}$ be a defining set of 1-forms for $\mathcal{D}$.

Lemma 15.4 (Lee, 19.6)**.** $\eta \in I(D)$ *if and only if* $\eta = \sum_{i=1}^{n-k} \omega^i \wedge \beta^i$ *for some forms* β^i.

Proof. $\Rightarrow$ At each point, we must make a local argument. Expand the coframe $\{\omega^i\}_{i=1}^{n-k}$ to $\{\omega^i\}_{i=1}^{n}$, and choose a dual frame (E_i). Suppose η is a p-form. Then $\eta = \sum \eta_I \omega^I$. Then the last k fields E_i are a frame for D. After plugging the frame E_i into the annihilator condition, it follows that $\eta_I = 0$ when all indices in I are $> n-k$. So we get the result because the only non-zero have an index from the first $n-k$.

$\Leftarrow$ This is immediate, since $\omega^i \wedge \beta^i$ vanishes on any collection of vectors in $\mathcal{D}$.

□

Math 546 Manifolds Notes by Avi Levy

Lecture 16: May 5

Recall some definitions from last time:

1. **Local defining forms** for a distribution $\mathcal{D}$: linearly independent set $\{\omega^i\}_{i\le n-k}$ of 1-forms on an open U such that

$$\mathcal{D}_g = \bigcap_{j\le n-k} \ker \omega^j_w, \qquad \forall q \in U$$

2. **Annihilator of** $\mathcal{D}$: This is the set

$$I(\mathcal{D}) = \{\eta \in \Omega^*(M) : \eta \text{ annihilates } \mathcal{D}\}$$

Lemma 16.1 (Lee, 19.6)**.** *$\eta \in I(\mathcal{D})$ holds if and only if for any set of local defining forms $(\omega^i)_{i\le n-k}$ for $\mathcal{D}$ on U,*

$$\eta|_U = \sum_{i\le n-k} \omega^i \wedge \beta^i \text{ for some } \beta^i \in \Omega^*(U).$$

Proof of →: $I(\mathcal{D})$ is an ideal in $\Omega^*(M)$; this means $I(\mathcal{D}) \wedge \eta \subset I(\mathcal{D})$ for any form η. □

Proposition 16.2 (Lee, 19.7)**.** *$\mathcal{D}$ is involutive precisely when*

$$\{\text{if } \eta \text{ is a 1-form in } I(\mathcal{D}), \text{ then } d\eta \in I(\mathcal{D})\}$$

Proof. Write $d\eta(X,Y) = X(\eta(Y) - Y(\eta(X)) - \eta([X,Y])$. Then for $X, Y \in \Gamma(\mathcal{D})$ and $\eta \in I(\mathcal{D})$, this reduces to $d\eta(X,Y) = -\eta([X,Y])$.

- ⇒ If $\mathcal{D}$ is involutive, then $[X,Y] \in \Gamma(\mathcal{D})$, so right side is 0.
- ⇐ In particular, it is true for defining 1-forms $X, Y \in \Gamma(\mathcal{D})$. Then $0 = d\omega^i(X,Y) = \omega^i([X,Y])$ for $i \le n-k$, so $[X,Y]$ takes values in $\mathcal{D}$.

□

Combining 19.6 and 19.7 yields the follows proposition:

Proposition 16.3 (Local coframe criterion for involutivity)**.** *Suppose that $(\omega^i)_{i\le n-k}$ are local defining forms. Then The following are equivalent:*

1. *$\mathcal{D}$ is involutive (in domain of the frame)*
2. *$d\omega^i \in I(\mathcal{D})$ for $i \le n-k$*
3. *$d\omega^i = \sum_{j\le n-k} \omega^j \wedge \alpha^i_j$ for smooth 1-forms α^i_j.*

Definition 16.4. An ideal $I \subset \omega^*(M)$ is called **differential** when $dI \subset I$

Proposition 16.5 (Lee, 19.11)**.** *$\mathcal{D}$ is involutive if and only if $I(\mathcal{D})$ is differential.*

Proof. ⇒ Use 19.6. Then for $\eta \in I(\mathcal{D})$, we can write $\eta = \sum \omega^i \wedge \beta^i$. Apply d, so that

$$d\eta = \sum d\omega^i \wedge \beta^i - \omega^i \wedge d\beta^i) \in I(\mathcal{D}),$$

where we have used the result of 19.8 and that $I(D)$ is an ideal.

⇐ This follows from 19.7, since the condition implies the result for 1-forms, so 19.7 applies.

□

Integral submanifolds and foliations

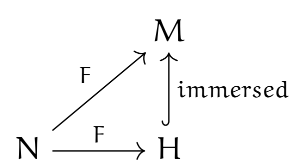
Recall the following definition:

Definition 16.6. An immersed submanifold $S \subset M$ is called **weakly embedded** (or **initial**) if for every smooth map $f: N \to M$ such that $F(N) \subset S$, the map $F: N \to S$ is smooth.

Proposition 16.7 (Lee, 19.17)**.** *An integral submanifold H of an involutive distribution on M is weakly embedded.*

Why H? This is anticipating theorem 19.25, which says that Lie subgroups are weakly embedded.

Proof. The key is to use flat charts for $\mathcal{D}$.



Initially, the bottom F is just a set map. We must show it is smooth. So take a flat chart (U, ϕ) and coordinates $x^1, \cdots, x^n$. Then $\langle \frac{\partial}{\partial x^i} \rangle$ span $\mathcal{D}$ at points in U. THen $H \cap U$ is contained in countable many slices where $x^{k+1}, \cdots, x^n$ are constant.

Consider $p \in F^{-1}(U) = F^{-1}(U \cap H)$. Take a connected neighborhood V of p. Then $F(V)$ is contained in a single slice. Near p, F^j for $j > k$ is constant. Hence $(F^1, \cdots, F^k)$ give a smooth composition of F as a map into H. □

Extending from local to global

We want to extend integral submanifolds to maximal connected domains. The collection of all such maximal submanifolds turns out to be a **foliation**.

Definition 16.8. A **foliation** of M is a partition of M into connected, immersed k-submanifolds (k fixed), called the **leaves** of the foliation, such that for every $p \in M$, there is a coordinate chart at p that is a flat chart for the foliation.

(U, ϕ) is a flat chart for a foliation if each $(x^1, \cdots, x^k)$ slice in the chart is contained in a single leaf.

Proposition 16.9 (Lee, 19.19)**.** *The set of tangent vectors to a foliation is an integral distribution.*

Conversely, given an involutive distribution D, given a point $p \in M$ let's consider a subspace

$$N = \bigcup_{\alpha} N_\alpha, \qquad N_\alpha \text{ is a connected integral submanifold of } D \text{ containing } p.$$

Locally, this looks like a manifold (since it gets its structure from the N_α). But is it too big?

Lemma 16.10 (It can't get too big, 19.22)**.** N *has a unique manifold structure that makes it an integral submanifold of* D.

Theorem 16.11 (Global Frobenius)**.** *The collection of all maximal connected integral submanifolds of an involutive distribution is a foliation.*

Easy parts of proof of lemma. For uniqueness, we use a result from chapter 5. If we get any such structure, it is unique by weakly embedded.

We give it the minimal coherent topology with respect to the N_α. Concretely,

$$U \text{ is open if and only if } U \cap N_\alpha \text{ is open for every } \alpha.$$

In other words, check $N_\alpha \cap N_\beta$ is open in both N_α and N_β. We're working in an integral distribution, so we work in a flat chart for it (ultimately, it will be a flat chart for the foliation). The rest of the properties of a topology follow routinely).

To give it a smooth structure, we use slices in flat charts again. Check smooth compatibiility, immersion into M and integral submanifold of $\mathcal{D}$. □

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Lecture 17: May 7

Technical remark on flat charts: they are defined so that the image $\phi(U)$ is a cube in $\mathbb{R}^n$. You want the slices to be connected pieces; the image should be a product of a connected set in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, and the proofs go through in this generality. But it's just easier to work with cubes.

We were proving lemma 19.22. Given D involutive, let $N = \cup N_\alpha$, where each N_α is a connected integral submanifold of D containing a given point p. The remaining issue was to prove that N is second countable.

Pick a countable cover of M by flat charts (W_i, ϕ_i) for D. Then $N \cap W_i$ is a union of slices of W_i. Since $q \in N$, it is in $q \in N_\alpha$ for some α. Then the entire slice has to be in N. We will get a countable number of slices, a countable number in each W_i, and then a countable number of W_i. Each slice is second countable, so we get the result.

Pick a slice S of a W_i such that S is part of N. Pick $q \in S$, so then $q \in N$ and hence $q \in N_\alpha$ for some α. Then there is a piecewise smooth curve $\gamma : [0,1] \to N_\alpha$ connecting p to q. Then $\gamma^{-1}(W_i)$ gives us an open cover of $[0,1]$, so there is a finite subcover. Then $\gamma([t_{j-1}, t_j]) \subset W_{i_j}$. Since the curve goes into N_α, its tangent space is in the distribution. Thus $\gamma' \in TN_\alpha$ which is the tangent space to the $x^1, \cdots, x^k$ slice. In other words, the last $n-k$ components of γ' are zero. When you look at the image of this interval, it is entirely contained in some chart, S_j.

So we have produced the following. For any given slcie S in N, a finite sequence of overlapping slices $S_1, \cdots, S_m = S$ (without loss of generality) connecting p to S. We do this for every slice in N. Count the slices in all of these sequences, by length of the sequence.

At length m, you have a countable number of choices for S_1. Then S_1 is a manifold. Now for any i, $S_1 \cap W_i$ is a union of open sets, each in 1 slice of W_i. Now for S_2, there are a countable number of slices in each W_i, and a countable number of W_i choices. So there are a countable number of choices for S_2. Similarly, there are a countable number of choices for $S_3, \cdots, S_n$. Taking all the choices into account, there are a countable number of total choices in determining a sequence of length m. So the total number of choices is a "sum" of the countable number of choices at stage n; this is still countable. Hence there are only countable many slices S_i that are accessible from p. Thus we end up with a second countable space.

Comments about the leaf space/quotient topology

Even though we haven't yet talked about smooth quotient manifolds, we can say some things about the quotient topology being badly behaved. Essentially, the "leaf space" $(M/\mathcal{F})$ may be really nasty. For example, consider the dense line on the torus; then D is spanned by $\frac{\partial}{\partial\theta} + \alpha\frac{\partial}{\partial\phi}$, for α irrational. By Frobenius there is a local solution, but in general you wouldn't expect to get a global solution. The space is **very** non-Hausdorff.

In Example 19.23 (f), we have the secant curves rotated around. In the leaf space, you can't separate the vertical lines by open sets; so it is non-Hausdorff, but in a "less-bad" way than the dense torus line.

Applications to Lie groups

Let G be the Lie algebra. Use global frobenius theorem to complete the proof that there is a bijection between Lie subalgebras and connected Lie subgroups of G. By theorem 8.46, each connected Lie subgroup H of G leads to the Lie subalgebra T_eH (the left invariant vector fields on H).

Theorem 17.1 (Lee, 19.26)**.** *Given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a Lie group G, there is a unique connected Lie subgroup $H \subset G$ whose Lie algebra is $\mathfrak{h}$.*

Proof. The subalgebra gives a smooth distribution (take all the vectors at the identity and translate them everywhere). Now we show that this distribution is involutive. Pick a basis of $\mathfrak{h}$; then it's a frame for $\mathcal{D}$ that is closed under brackets. Let H be the maximal connected integral submanifold of $\mathcal{D}$ through e.

Now we cite the following proposition:

Proposition 17.2 (Lee, 19.23)**.** *If $\phi\colon M \to M$, then an involutive distribution $\mathcal{D}$ is invariant under ϕ if and only if the leaves of the foliation are invariant.*

After the proposition, we make two short computations to show that H is closed under multiplication and inversion. So we have a group.

To show smoothness of the group operations, start with $i\colon H \to G$. Since its image lies in H and H is weakly embedded, we can restrict the codomain to obtain a smooth map. The same argument works for m, so we actually have a Lie subgroup.

Now for uniqueness, observe that any connected subgroup $\tilde{H}$ with this Lie algebra is a connected integral submanifold of D. By maximality of H, it follows that $\tilde{H} \subset H$. Working on a neighborhood of the identity (in our flat chart), then $\tilde{H}$ contains the neighborhood of the identity in H. Hence $\tilde{H}$ contains a neighborhood of e in H. For topological group, this was a test problem from fall quarter. But if you look at Proposition 7.14, it follows that $\tilde{H} = H$. □

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Lecture 18: May 9

PDE's

Homogeneous PDEs. Find u such that $Xu = 0, Yu = 0E$, if you're given u on a submanifold "complementary" to a foliation. We discussed this for the example used in the proof of Frobenius. At the end of the example, we had $(0, 1, c)$ for $c \in \mathbb{R}$.

Also recall Example 19.1 (d), which was our first non-integrable distribution:

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$$
$$Y = \frac{\partial}{\partial y}$$

We showed that there is a piecewise smooth γ such that $\gamma'(t) \in D$ for all t, and connects $(0, 0, 0)$ to any point.

There's a related homework problem 19-6; if there were a leaf for this folation, it would have to include every point in $\mathbb{R}^3$; points joined in this manner must belong to a single leaf if the distribution is integrable.

If the distribution is integrable, you can count dimensions. Otherwise, it might be that you have less freedom than you think. In this case, the only solutions to $Xu = Yu = 0$ are constant; this is less than the $n - \dim \mathcal{D}$ "degrees of freedom".

Proposition 18.1 (Lee, 19.27). *On $\mathbb{R}^n$, suppose that $Xu = f_1$ and $Yu = f_2$. Then we need $[X, Y]u = Xf_2 - Yf_1$; moreover, if those conditions are satisfied, then this is an integrable distribution. Hence you can find the commuting coordinates and use them to solve the differential equation.*

Proposition 18.2 (Lee, 19.28, 19.29). *The idea is to find the graph of the solution as an integral submanifold of a distribution on a higher dimensional space.*

This has an application to Riemannian geometry; this is the method of solution (i.e., saying that a distribution is involutive). In the second volume of Spivak's six volume series, he goes over six proofs of this. Debauche indices, moving frame, Koszul connection, Levi-Civita, and then Frobenius.

Example 18.3.

$$\frac{\partial u}{\partial x^1} = \alpha_1(x^1, x^2, u(x^2, x^2))$$
$$\frac{\partial u}{\partial x^2} = \alpha_2(x^1, x^2, u(x^2, x^2))$$

Englarge to three variables (x^1, x^2, u); make a distribution on $\mathbb{R}^3$ whose integral submanifolds are graphs of solutions to our original problem. We need the mixed partials to agree, which leads to the conditions

$$\frac{\partial \alpha_1}{\partial x^2} + \frac{\partial \alpha_1}{\partial u}\alpha_2 = \frac{\partial \alpha_2}{\partial x^1} + \frac{\partial \alpha_2}{\partial u}\alpha_1.$$

Then we construct a distrubtion from $\frac{\partial}{\partial x^i} + \alpha_i \frac{\partial}{\partial x^i}$. The naive necessary condition turns out to exactly be the statement that the distribution you get up in $\mathbb{R}^3$ is involutive.

That's the overall idea of the Frobenius Theorem: you put the naive constraint on solutions, and it turns out that that is sufficient!

The exponential map

Recall from the 545 final that we had a map

$$\exp\colon M(n, \mathbb{R}) \simeq \mathfrak{gl}(n, \mathbb{R}) \to M(n, \mathbb{R}).$$

Initially, this was a map on the same space:

$$\exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

You can show that the image $\subset Gl(n, \mathbb{R})$, because $e^A e^{-A} = \mathrm{Id}$ (since $\|A\| < \infty$). Similarly, $e^{sA}e^{tA} = e^{(s+t)A}$.

Proposition 18.4 (Lee, 20.2)**.** *The map*

$$t \mapsto e^{tA}$$

is a Lie group homomorphism into $GL(n, \mathbb{R})$.

Very rough. We use uniform convergence of e^A for bounded A. Next, e^{tA} is an integral curve for the left-invariant vector field corresponding to $A \in T_e GL(n, \mathbb{R}) \simeq M(n, \mathbb{R})$. Then we show $\frac{\partial}{\partial t}\left(e^{tA}e^{-tA}\right) = 0$, and from there we get an inverse. □

Definition 18.5. A **one-parameter subgroup** is a Lie group homomorphism $\gamma\colon \mathbb{R} \to G$.

Note that a one-parameter subgroup is not a Lie group, but its image is a Lie subgroup of G.

Here's out plan: For each $X \in T_e G/\mathfrak{g}$, we get a one-parameter subgroup γ_X as an integral curve of the left invariant vector field X such that $\gamma_X(0) = e$. Then we define $\exp(X) = \gamma_X(1)$. This is a *very pointwise* construction; at first, it feels like there's no hope for it to be smooth.

However, we show that exp is indeed a smooth map:

$$\exp\colon T_e G \to G, \gamma_X(t) = \exp(tX).$$

There are lots of other nice properties. It even turns out that the right multiplication by t gives you the flow of X!

Theorem 18.6 (Characterization of one-parameter subgroups, 20.1)**.** *They are the integral curves of a left-invariant vector field.*

Proof. Given a maximal integral curve $\gamma_X(t)$ of a left invariant vector field on X, show it's a Lie group homomorphism. By Theorem 9.18, X is complete. Let θ be the flow of X. Then $L_g \circ \theta_t = \theta_t \circ L_g$. Consequently

$$\begin{aligned}
\gamma_X(s) \cdot \gamma_X(t) &= \theta_s(e) \cdot \theta_t(e) \\
&= L_{\theta_s(t)} \theta_t(e) \\
&= \theta_t \left(L_{\theta_s(e)}(e) \right) \\
&= \theta_t \left(\theta_s(e) \right) \\
&= \theta_{t+s}(e) \\
&= \gamma_X(s+t).
\end{aligned}$$

Conversely, suppose that $F\colon \mathbb{R} \to G$ is a one parameter subgroup. Now consider the field $\frac{\partial}{\partial t}$ on $\mathbb{R}$; it is left-invariant, so $dF\left(\frac{\partial}{\partial t}\right)$ is left invariant on its image. Extend to a left invariant vector field such that $\gamma_* \frac{\partial}{\partial t} = X$, so γ is an integral curve of X. □

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Lecture 19: May 12

Lie algebras and Lie groups have lots of structure, and therefore lots of identifications - which can be confusing. As before, 1-psg is shorthand for a one-parameter subgroup.

$$T_eG \xleftrightarrow{\text{v. space iso.}} \mathrm{Lie}(G) \subset \mathfrak{X}(M)$$

$T_eG \leftrightarrow \mathrm{Lie}(G) \subset \mathfrak{X}(M)$. If you have a left invariant vector field X, you can identify it with X_e. She may write just X for X_e (especially in this context).

Also $\mathrm{Lie}(G) = \mathfrak{g}$ is **not** an identification, it is just two different notations for the same thing.

Definition 19.1. $\exp\colon \mathfrak{g} \to G$, given by $X \mapsto \gamma(1) = \exp X$.

exp is often thought of as a map from the tangent space at the identity into the group.

For $X \in \mathfrak{g}$, let γ be the maximal integral curve with $\gamma(0) = e$. Then γ is a 1-psg corresponding to X.

Also used in other areas, for instance in Riemannian geometry to get from a vector to a geodesic. If you pick the right metric on a group, then it turns out to be the same thing. All came from exponentiating matrices, which came from solving constant coefficient ODEs.

Every vector space is a manifold. So we can consider the differential of the map $\exp\colon T_eG \to G$:

$$d(\exp)_0\colon T_0(T_eG) \to T_eG.$$

But $T_vV \simeq V$ for any vector space (since vector spaces are flat). Hence $T_0(T_eG) \simeq T_eG$, as we'll see in a bit.

Lemma 19.2 (Lee, 20.5)**.** *$\exp(sX)$ is the 1-psg $\gamma(s)$ generated by X.*

Proof. Let θ be the flow of X, so $\theta(t, e)$ is a 1-psg for X. Similarly, $\tilde{\theta}$ is the flow of sX, so $\tilde{\theta}(t, e)$ is a 1-psg for sX.

By the Rescaling Lemma (9.3), $\tilde{\theta}(t, e) = \theta(st, e)$. Evaluating at $t = 1$ yields $\exp(sX) = \gamma(s)$ (the integral curve of X starting at e). □

Let $\theta_{(X)}(t, g)$ be the flow of X. We also write this as $\theta^{(g)}_{(X)}(t)$, thought of as an integral curve in t.

Proposition 19.3. *(a)* $\exp\colon \mathfrak{g} \simeq T_eG \to G$ *is a smooth map*

(b) $\exp((s + t)X) = \exp(sX)\exp(tX)$

(c) $(\exp X)^{-1} = \exp(-X)$

(d) $(\exp X)^n = \exp(nX)$

(e) $(d(\exp))_0 = \mathrm{Id}\colon T_g\mathfrak{g} \simeq T_0(T_eG) \simeq T_eG \to T_eG$

(f) $\exp$ *is a local diffeomorphism at the origin; in other words,* $\exp$ *is a diffeomorphism from some neighborhood of 0 in* T_eG *to some neighborhood of* e *in* G*.*

(g) If Φ *is a Lie group homomorphism* $\Phi\colon G \to H$*, then we have a commuting diagram*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Phi_*} & \mathfrak{h} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow[\Phi]{} & H \end{array}$$

(h) $\theta_{(X)}(t,g) = R_{\exp(tx)}g$

Proof. Note: (b) and (c) are immediate from Lemam 20.5; they imply (d). The Rank Theorem and (e) imply (f).

For (a), we use the same trick as in the proof of 8.37 (showing that left invariant vector fields are smooth). The idea is to enlarge your space, get an obviously smooth operator on the larger space, and get one of its derivatives to be the object under consideration.

So work in $G \times \mathfrak{g}$. The vector field tangent to each slice is $G \times \{X\}$, which is X on that copy of G. So we define a vector field, Ξ, on $G \times \mathfrak{g}$. We show its smooth by acting on smooth functions $f \in C^\infty(G \times \mathfrak{g})$.

Now since Ξ is smooth and all flows on a Lie group are complete, it has a global flow $\Theta\colon \mathbb{R} \times G \times \mathfrak{g} \to G \times \mathfrak{g}$. We claim that

$$\Theta(t,g,x) = (\theta_{(X)}(t,g), X).$$

So we compute that

$$\left.\frac{d}{dt}\right|_0 \big(\theta_{(X)}(t,g), X\big) = (X_g, 0) = \Xi(g, X_g).$$

Then we check that

$$\exp(X) = \pi_1 \circ \Theta|_{\{1\}\times\{e\}\times G}$$

(π_1 since we're projecting down to G). It follows that $\exp(X)$ is smooth, proving part (a).

Next we prove (e); for clarity, let $\tilde{X} \in T_0(T_eG)$ correspond to $X_e \in T_eG$ and $X \in \mathfrak{g}$.

$$\begin{array}{ccc} T_0(T_eG) & \xrightarrow{d(\exp)_0} & T_eG \\ \downarrow & & \downarrow \\ T_eG & \xrightarrow{\exp} & G \end{array}$$

In the absence of coordinates, the best way to work with differentials is with curves. So we should represent $\tilde{X}$ by a curve tX (i.e., line through origin). Now we exponentiate the curve.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{d(\exp)_0} & X \\ \downarrow & & \uparrow \frac{\partial}{\partial t} \\ tX & \xrightarrow{\exp} & \exp(tX) \end{array}$$

But $\tilde{X}$ was just a way to think of X under the right identification; so we get the identity map as desired.

Now for the proof of (g), we (very slightly) generalize the identity in (e). So we establish $\exp(t\Phi_* X) = \Phi(\exp tX)$.

Indeed, $t \mapsto \Phi(\exp tX)$ is a 1-psg of H. Then

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_{t=0} \Phi(\exp tX) &= d\Phi_e\left(\frac{\partial}{\partial t}\exp tX\right) \\ &= d\Phi_e(X) \\ &= \Phi_*(X). \end{aligned}$$

So the 1-psg corresponds to $\Phi_*(X) \in \mathfrak{h}$, so it is $\exp t\Phi_* X$, completing the proof of (g).

The result of (h) expresses the annoying fact that left and right actions are completely intertwined on Lie groups. We compute

$$\begin{aligned} R_{\exp tX} g &= g \exp tX \\ &= L_g(\exp tX) \\ &= L_g(\theta_{(X)}(t, e)) \\ &= \theta_{(X)}(t, L_g(e)) \qquad \text{(Naturality of Integral Curves, 9.6)} \\ &= \theta_{(X)}(t, g). \end{aligned}$$

□

This proves (h).

Closed Subgroup Theorem

This is theorem 20.12. Given a topologically closed algebraic subgroup $H \subset G$ where G is a Lie group, we in fact have that H is an embedded Lie subgroup. As a corollary, an algebraic subgroup is topological closed if and only if it is an embedded submanifold if and only if it is an embedded Lie subgroup.

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Lecture 20: May 14

Last time, we stated the closed subgroup theorem and its corollary (20.13). To prove the theorem, we start by proving Corollary 20.11. It says that locally, our flows almost commute; in other words,

$$\lim_{n\to\infty}\left[\left(\exp\frac{t}{n}X\right)\left(\exp\frac{t}{n}Y\right)\right]^n = \exp t(X+Y).$$

This follows from the (second order approximation of) the Baker-Campbell-Hausdorff formula:

$$(\exp tX)(\exp tY) = \exp t(X+Y) + O(t^2).$$

For instance, see problem 20-9; the key idea in its proof is Problem 7-2, which says that the derivative of multiplication is addition. This gives the result (after essentially expanding a low-order Taylor series).

Outline of proof. 1. We identify a candidate for $\mathrm{Lie}(H) \subset \mathrm{Lie}(G)$ (which we call $\mathfrak{h}$).

2. We show that there's a neighborhood U of $0 \in \mathfrak{g}$ such that

$$\exp(U \cap \mathfrak{h}) = H \cap \exp(U).$$

The picture is: we have a neighborhood of 0 sliced by a line, and exp maps it into a curve through e in $\exp(U) \subset G$.

3. We get a flat chart by pulling back $\mathfrak{h}$ along $\exp^{-1}$. This gives a slice chart for H at e; then we left translate by $h \in H$ to get the slice chart for H at h.

Overview: Steps 1 and 2 are particular to this proof, so we won't dwell on them too much. But Step 3 is a general technique that we will use on similar proofs. In more detail:

1. Define the subspace

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H, \qquad \forall t \in \mathbb{R}\}.$$

To show it is a linear subspace, we use that H is a subgroup, H it is topologically closed, and then the lemma.

2. This is the major step. Suppose there is no such U. Then there is a sequence of points $z_i \to 0$ with $\exp z_i = h_i$. Split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Write $Z_i = X_i + Y_i$ and obtain the contradiction.

Pick an arbitrary inner product and normalize, so we're looking at $\frac{1}{|Y_i|}Y_i$. Some clever analysis shows that because H is close,d $\exp(tY) \in H$ for all t. Then by definition of $\mathfrak{h}$, $Y \in \mathfrak{h}$.

3. We can use $\exp^{-1}$ to get slice coordinate charts on U. Then we left translate to get a slice chart elsewhere.

□

Infinitesimal Generators

We have already encountered two cases; note that the flow of a vector field is an $\mathbb{R}$-action.

Group Action	Infinitesimal Generator	Explanation
Flow of a complete v.f. X	X	$\alpha\frac{\partial}{\partial t} \mapsto \alpha X$
Right G-action acting on itself	$\mathrm{Lie}(G) \mapsto \mathfrak{X}(G)$	identity map (identifications)

We are implicitly using 20.8(h), which says that the flow $\theta_{(X)}$ for a left invariant vector field on X is $\theta_t = R_{\exp(tX)}$. We will be interested in right actions (also in the next chapter about homogeneous spaces). There is this interplay between left actions and right actions that is unfortunately baked in.

For a right action of G on M,

$$\begin{aligned} M \times G &\longrightarrow M \\ (p, g) &\longmapsto p \cdot g. \end{aligned}$$

Sometimes we will omit the $\cdot$

Definition 20.1. The **infinitesimal generator** of θ is

$$\hat{\theta}\colon \mathrm{Lig}(G) \to \mathfrak{X}(M), \qquad \hat{\theta}(X)\big|_p = \hat{X}_p = \frac{d}{dt}\left(p \cdot \exp(tX)\right).$$

In the book, only the map is the infinitesimal generator. However, it's common to also say that $\hat{X}$ **is the infinitesimal generator of** θ **associated to** X.

Here's another perspective on $\hat{\theta}$; we use the orbit map. Consider a curve in G through e (for instance, a 1-psg); under the action of the orbit map $\theta^{(p)}$, it gets sent to the orbit $p \cdot G$. This is usually a proper subset of M. If the action is free, then we showed thta the image is a submanifold. It turns out (using the results of chapter 21) that $p \cdot G$ is **always** a submanifold.

The infinitesimal generator of $\hat{\theta}$ takes each vector in the tangent space at the identity to a vector at $p \in M$. So they will consist of the tangents to the orbit. In other words,

$$d(\theta^{(p)})_e X_e = \hat{X}_p \qquad (\star)$$

Here $X \in \mathrm{Lie}(G)$, and therefore a left-invariant vector field on G.

The infinitesimal generator is a map from the Lie algebra over to vector fields on M, whose integral curves are the 1-psg through e. Alternatively, we think of the vector field associated to X as the infinitesimal generator (with respect to X).

Lemma 20.2 (Lee, 20.14)**.** *Generalization of* $(\star)$ *from* p *to all points* $p \cdot g$ *in orbit of* p.

Proof. First we write $\theta(p \cdot \{g \exp tX\}) = p \cdot \{g \exp tX\}$. Now take the derivative of both sides at $t = 0$; consequently

$$\begin{aligned} \frac{\partial}{\partial t}\theta(p \cdot \{g \exp tX\})\Big|_{t=0} &= \frac{\partial}{\partial t} p \cdot \{g \exp tX\}\Big|_{t=0} \\ &= \frac{\partial}{\partial t}(p \cdot g) \exp tX\Big|_{t=0} \\ &= \hat{X}_{p \cdot g}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{\partial}{\partial t}\theta(p \cdot \{g \exp tX\})\Big|_{t=0} &= \frac{\partial}{\partial t}\Big|_{t=0} \theta^{(p)}(L_g \exp tX) \\ &= (d\theta^{(p)})_{L_g(e)} dL_g(X_e) \\ &= d\theta^{(p)}_g X_g. \end{aligned}$$

Consequently $d\theta^{(p)}_g X_g = \hat{X}_{p \cdot g}$. □

Recall that $p \in M$ was arbitrary, so we have showed that X is $\theta^{(p)}$-related to $\hat{X}$. Then by naturality of the Lie bracket, $[\hat{X}, \hat{Y}]_p = \widehat{[X, Y]}_p$. It's like we're creating a bunch of partial differential equations, for which the group action is a solution. This is why Frobenius is about to rear its head again! The punchline of all this is that we have a Lie algebra homomorphism.

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Lecture 21: May 16

From last time, we had an infinitesimal generator $\hat{\theta}\colon \mathrm{Lie}(G) \to \mathfrak{X}(M)$, given by $X \mapsto \hat{X}$.

$$\hat{X}_p = \frac{d}{dt}\bigg|_{t=0} p \cdot \exp(tX) = d\theta_e^{(p)} X_e.$$

Intuitively, we think of it like "$\frac{\partial}{\partial g}\big|_e p \cdot g$".

More generally, $d\theta_g^{(p)} X_g = \hat{X}_{p \cdot g}$. Then by naturality, we obtain $[\hat{X}, \hat{Y}]_p = \widehat{[X, Y]}_p$. This implies the following theorem:

Theorem 21.1 (Lee, 20.15)**.** $\hat{\theta}\colon \mathrm{Lie}(G) \to \mathfrak{X}(M)$ *is a Lie group homomorphism.*

With the left action, it is an anti-homomorphism. Given a Lie algebra homomorphism $\hat{\theta}\colon \mathrm{Lie}(G) \to \mathfrak{X}(M)$ such that every $\hat{X}$ is complete, is there a group action θ for which $\hat{\theta}$ is its infinitesimal generator?

Let's see what could go wrong. First, consider what happens when exp is no injective. Then $\exp X = \exp Y$ for $X \neq Y$. Then $\theta(p, \exp tX)$ is the integral curve of $\hat{X}$ starting at p. We want it to equal $\theta(p, \exp tY)$. Setting $t = 1$, they had better agree; but there is no particular reason for this to happen!

For example, consider the circle $G = \mathbb{S}^1$ and $\mathfrak{g} = \mathbb{R}$. Take $\hat{X} = \frac{\partial}{\partial x}$ on $M = \mathbb{R}^2$. Then $\hat{\theta}\colon \frac{d}{dt} \mapsto \hat{X} = \frac{\partial}{\partial x}$. Think of $X = \frac{d}{dt}$ as a basis for $T_1\mathbb{S}^1$. Then we get a contradiction, because we can flow around π or $-\pi$ units and end up in the same place, but in $\mathbb{R}^2$ we end up moving in two different directions.

Theorem 21.2 (Fundamental Theorem of Lie Algebra Actions)**.** *Given a simply connected Lie group* G *and a Lie algebra homomorphism* $\hat{\theta}\colon \mathrm{Lie}(G) \to \mathfrak{X}(M)$ *such that every* $\hat{X} \in \mathrm{Im}(\hat{\theta})$ *is complete, there is a unique smooth right* G*-action* θ *on* M *with infinitesimal generator* $\hat{\theta}$*.*

Outline of proof. The idea is to construct an orbit map $\theta^{(p)}(g)$ by finding its graph as the leaf of a foliation for an integrable distribution. Then we check the result is a group action, smooth, and is unique.

On $G \times M$, consider the distribution

$$\{(X_g, \widehat{X}_p)_{(g,p)}, \qquad \forall X \in \mathfrak{g}, p \in M, g \in G\}.$$

Since it's a Lie algebra homomorphism, the distribution is involutive. Now we show that the leaf S_p through (e, p) is the graph of a function from G to M. We will show that $\pi_p\colon S_p \to G$ is a surjective submersion, find its local sections, and show that it is evenly covered. Hence π_p is a covering map of the simply connected space G; thus it is a diffeomorphism (by universality of simply connected spaces).

Pick a neighborhood U of $0 \in T_eG$ such that $\exp|_U : U \simeq V$ for a neighborhood of $e \in G$. Now consider $L_g : (e, V) \to (g, gV)$, with local section $g\exp(X)$. Then we get $(g\exp(X), \eta_{\widehat{X}}(1, q))$ (you flow the expected amount to get it to work). The last step is to show that no two such sections intersect (unless they coincide; you first show the x has to be the same, then use backwords flow to get the same q; we've done this argument on past homework problems). Thus we proceed to the last step, and use universality to conclude its a diffeomorphism.

Define $\theta^{(p)}(g)$ to be the point in S_p above g; that is, if $(g, q) \in S_p$ then $p \cdot g = q$. We need to check that it's a G-action. The interesting part is showing that it's multiplicative (which amounts to saying that $S_{p\cdot g}$ is sort of a translate of S_p). The trick is to consider $\Psi_g(g', r) = (gg', r)$. Then Ψ_g leaves the distribution invariant, so $d\Psi_g : (\widehat{X}_{g'}, X_r) \mapsto (\widehat{X}_{gg}, X_r)$. Hence $\Psi_g(e, p \cdot g) = (g, p \cdot g)$. But $(e, p \cdot g) \in S_{p\cdot g}$, and similarly fro S_p. Since Ψ leaves the leaves invariant, $\Psi_g : S_{p\cdot g} \to S_p$.

It follows that $(g', r) \in S_{p\cdot g}$ if and only if $(gg', r) \in S_p$. But then $(p \cdot g) \cdot g' = r = p \cdot (gg')$, as desired. To show it is smooth, consider $\exp|_U : U \simeq V$. Let $l = (\exp|_U)^{-1}$. Then $M \times V \simeq M \times U \subset M \times \mathfrak{g}$, where the vector field $\Xi(p, X) = (\widehat{X}_p, 0)$ (there is a typo in the book that says $\widehat{X}_g$; the g is incorrect and should be p). The flow is

$$(\eta_{\widehat{X}}(t,p), X) : \quad \mathbb{R} \times M \times V \simeq \mathbb{R} \times M \times U \subset \mathbb{R} \times M \times \mathfrak{g} \longrightarrow M \times \mathfrak{g}$$

$$(t, p, g) \longmapsto (\eta_{\widehat{l(g)}}(t, p), X).$$

Then follow the same argument for smoothness as before; restrict to $t = 1$ and project down to M, so $(p, g) \mapsto \eta_{\widehat{l(g)}}(t, p) = p \cdot g$ (which is smooth on V). Now apply the general principle that Lie groups look the same everywhere; so translate from the origin and we're done. □

Math 546 Manifolds Notes by Avi Levy

Lecture 22: May 19

Last time, we proved the Fundamental Theorem of Lie Algebra actions [Lee, 20.16] Given $\hat{\theta}\colon \mathrm{Lie}(G) \to \mathfrak{X}(M)$ with all $\hat{X}$ complete and G simply connected, then there is an action θ that induces it.

As a consequence, we have Theorem 20.19: If G is simply connected, then every Lie algebra homomorphism $\phi\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is induced by a Lie group homomorphism $\Phi\colon G \to H$.

Then there is the Lie correspondence (20.21) between isomorphism classes of finite dimensional Lie algebras and isomorphism classes of simply connected Lie groups. She won't talk about this nor the Normal subgroup/Ideal correspondence.

Local Lie groups

Local Lie groups are like flows for incomplete vector fields (where the group is $\mathbb{R}$). This is included in the text for historical interest. When mathematicians started getting the ideal of manifolds, it was with Riemann surfaces in complex variables. Local Lie groups are not directly studied currently.

The "spirit" of Local Lie groups appears in the study of PDEs: the "semigroups" (wait a minute, she just described groupoids - the multiplication is only partially defined).

The Adjoint Representation

G acts on itself by conjugation: $C_g\colon h \to ghg^{-1}$. This fixes e, so $(dC_g)_e\colon T_eG \to T_eG$ is a map from $\mathfrak{g} \to \mathfrak{g}$. Hence we have a representation of G on its Lie algebra.

$\mathrm{Ad}\colon G \to GL(\mathfrak{g})$. $C_A(B) = ABA^{-1}$ which is linear in B, so $(dC_A)_eY = AYA^{-1}$. Differentiating "with respect to g" (so to speak),

$$d(\mathrm{Ad})_e = \mathrm{Ad}_*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

(lots of g's, lots of structure)

For a matrix group, what even is this?

$$d(\mathrm{Ad})_e(X)(Y) = \frac{d}{dt}\bigg|_{t=0} (\exp tX)Y(\exp -tX)$$

for $A \in G$, $Y \in \mathrm{Lie}(G) \simeq T_eG$, and the thing we're differentiating is just $\mathrm{Ad}_{\exp tX}Y$. One can take as the definition of ad

$$\mathrm{ad}(X)(Y) = (d\,\mathrm{Ad})_e(X)(Y) = [X, Y]$$

The moral of the story is that if you forget something, do it for matrix groups where the computations are easy (then it will hold on all Lie groups with suitable generalization).

There are **two** adjoin maps! Don't "confabulate" them together (her words).

Theorem 22.1 (Lee, 20.27)**.** *Extend the last result on* ad *to all Lie groups.*

Proof. The flow of X is $R_{\exp tX}$ and right/left translation commute. □

Lastly the map $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $X \mapsto [X, \cdot]$ is a Lie algebra homomorphism. Either do this using 20.27, or observe that it's just the Jacobi identity. Nothing else to say here, moving on to the next chapter!

Chapter 21

Assume G is a Lie group acting continuously on a manifold M. Recall that M is locally compact Hausdorff; for most results, we won't need smooth.

Overview of main results:

Theorem 22.2 (Quotient Manifold Theorem, 21.10)**.** *Given a free proper action on a smooth manifold* M*, then* M/G *is a topological manifold of dimension* $\dim M - \dim G$ *and has a unique smooth structure such that* $\pi\colon M \to M/G$ *is smooth.*

Now two results on homomogeneous spaces (these are smooth manifolds M with a smooth, transitive G-action.

Theorem 22.3 (Lee, 21.17)**.** *If* H *is a closed subgroup of* G*, then we get the results of 21.10 for* G/H*. Furthermore, a left* G*-action on itself descends to* G/H *(making it a homogeneous space).*

Theorem 22.4 (Lee, 21.18)**.** *Every homogeneous space is of the above form.*

Proof. Pick $p \in M$, and consider the map $G/G_p \to M$ given by $g \cdot G_p \mapsto g \cdot p$. □

Big result 3: Constructing homogeneous spaces from a set X with a transitive group structure.

Proposition 22.5 (Lee, 21.20)**.** *If for some* $p \in X$, G_p *is closed, then there is a unique smooth manifold structure on* X *such that the action is smooth.*

Lastly, we talk about applications to Lie groups (projective space, grassmanians, and flag varieties).

Starting the chapter

The key tool for all this is the proper action. It answers the question: what condition for a group action on a locally compact hausdorff space X will make X/G Hausdorff?

Let's look at two examples of $\mathbb{R}$ acting on $\mathbb{R}^2$. There is translation: $t \cdot (x, y) = (x + t, y)$, and there is the shear: $t \cdot (x, y) = (x + yt, y)$.

The coset space of the first is just $\mathbb{R}$, whereas the second is emphatically NOT Hausdorff; it is the line with uncountably many origins!

Intersect with a compact set $\times\mathbb{R}$.

Math 546 Manifolds Notes by Avi Levy

Lecture 23: May 21

Last time we gave the following definition:

Definition 23.1. A continuous left action θ of a topological group on a topological manifold M is called **proper** if $\theta \times \mathrm{Id}$ is a proper map.

Proposition 23.2. *Consider a Lie group acting continuously on a manifold M. The following are equivalent:*

(a) θ *is proper*

(b) Given $p_i \to p$ *and* $g_i \cdot p_i \to q \in M$*, then* $\{g_i\}$ *has a convergent subsequence*

(c) Given compact $K \subset M$*, then*

$$\{g \in G \colon g \cdot K \cap K \neq \varnothing\} \text{ is compact.}$$

When people started thinking of proper, they formulated something similar to (c); indeed, it can be built up by starting with a single point (free) then finitely many points (so no bad convergence issues) and finally extending to compact.

Proof. Last quarter we proved that (a) and (c) are equivalent, but now we do it again.

(a) $\Longrightarrow$ (b) Consider a sequence $A = \{(g_i, p_i)\}$. Then $\theta \times \mathrm{Id}(A)$ converges, so eventually $\theta \times \mathrm{Id}(A) \subset \overline{V} \times \overline{U}$, where V, U are precompact neighborhoods of p, q. Hence the tail of A belongs to $(\theta \times \mathrm{Id})^{-1}(\overline{V} \times \overline{U})$ which is compact. Thus $g_i = \pi_1(g_i, p_i)$ is compact, so it has a convergent subsequence.

(b) $\Longrightarrow$ (c) Consider a sequence $\{g_i\} \subset G_K$. Then for each i, $p_i \in g_i \cap K$. Hence both p_i and $g_i^{-1} \cdot p_i \in K$. By compactness of K, $\{p_i\}$ has a convergent subsequence. Restrict attention to those indices, and consider the corresponding $g_i^{-1} \cdot p_i \in K$. In this manner we obtain a subsequence on which they both convergence; hence g_{n_i} converges, as desired.

(c) $\Longrightarrow$ (a) Given a compact $L \subset M \times N$, define the set $K = \pi_1(L) \cup \pi_2(L)$. Then $L \subset K \times K$ which is compact. Hence

$$(\theta \times \mathrm{Id})^{-1}(L) \subset (\theta \times \mathrm{Id})^{-1}(K \times K) \subset G_K \times K,$$

which is compact. Then $(\theta \times I)^{-1}(L)$ is a closed subset (by Hausdorff) of a compact set, and is therefore compact.

□

Corollary 23.3 (Lee, 21.6)**.** *A continuous action of a compact Lie group is proper.*

Corollary 23.4 (Lee, 21.8)**.** *If an action is proper, then all of its stabilizer subgroups* G_p *are compact.*

Lemma 23.5 (Lee, 21.1)**.** *The quotient map* $M \to M/G$ *is open.*

Proposition 23.6 (Lee, 21.4)**.** *If* G *acts continuously and properly on* M*, then* M/G *is Hausdorff.*

We omit the proofs because they are easy and we did them in ITM! Note that all these results hold in the topological category (and therefore in the smooth category).

Proposition 23.7 (Lee, 21.7)**.**

1. $\theta^{(p)}$ *is a proper map*
2. $G \cdot p = \operatorname{Im} \theta^{(p)}$ *is closed*
3. *If* $G_p = 0$*, then* $G \cdot p$ *is properly embedded in* G

Proof.

1. Given a compact $K \subset M$, $\theta^{(p)}(g) = g \cdot p \in K \Leftrightarrow g \in G_{K \cup \{p\}}$. By the (c) characterization of proper, $\theta^{(p)-1}(K)$ is a closed subset of a compact set, and hence compact.
2. Proper maps are closed
3. It is the image of a proper immersion, and hence by chapter 5 results it is properly embedded

□

Theorem 23.8 (Quotient Manifold Theorem, 21.10)**.** *Given a group* G *that acts smoothly, freely, and properly on* M*, then*

1. M/G *is a topological manifold of dimension* $n = \dim M - \dim G = m - k$
2. M/G *has a unique smooth structure such that* $\pi\colon M \to M/G$ *is a smooth submersion*

Proof. Uniqueness of smooth structure follows from uniqueness of smooth quotients. Now there are two topological items to prove:

1. Proper actions mean that M/G is Hausdorff
2. The quotient map is open, and a basis pushes forward to another basis (prove it); thus M/G is second countable

The image of $\hat{\theta}$ spans an integral distribution $\mathcal{D}$ on M, whose orbits are integral submanifolds.

Indeed, the free action $\hat{\theta}(\mathfrak{g})|_p = T_p(G \cdot p)$ has constant dimension k. Pick a (vector space) basis for $\mathfrak{g}$; then they push forward to a global frame for the subbundle $\mathcal{D}$ of M. Now those vector fields will be $\theta^{(p)}$-related to the vector fields on $\mathfrak{g}$, so their brackets will still vanish. Hence $\hat{\theta}$ is a Lie algebra homomorphism, so $\mathcal{D}$ is involutive, and therefore it is integrable by Frobenius.

Now fix $p \in M$ and choose a flat chart U around p. The k-slice through p would is entirely contained in $G \cdot p$ (by a connectedness argument). We want for every q, $U \cap (G \cdot g)$ is either empty or an entire k-slice. Then we could use the coordinates on Y as the coordinates on our quotient space. If this occurs, we call U a "G-adapted" chart.

We assume that U is G-adapted for now, because that is the hard step in the proof. Now use $(x^{k+1}, \cdots, x^m)$ as (local) coordinates for M/G; thus we get a topological manifold structure on M/G.

At last, we check that these local coordinate charts are smoothly compatible (slightly more tedious). □

A little word about the proof of the hard part (which we will do next time): We start from Y and flow out using the group action. We don't know that it's nice for the entire group, but we'll find a neighborhood X around $e \in G$ such that θ restricts to a diffeomorphism from $X \times Y$ to a neighborhood of Y in M.

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Lecture 24: May 23

Last time, we did most of the quotient manifold theorem (21.10). To finish the proof, it remains to show that every $p \in M$ there is an adapted chart. This means it is flat for the foliation by the connected components of the G-orbits, and $G \cdot q \cap U$ is "all or nothing" (either empty, or a single slice). This gives us our local Euclidean structure on M/G at $G \cdot p$.

Finishing the proof (hard part). The idea is to do a "flowout" from Y, where Y is a k-slice. Look at $d\theta_{(e,p)}$; restrict θ to $\{e\} \times Y$. When you restrict the flow, you obtain π_2. Hence

$$d\theta_{(e,p)} \colon \{0\} \times T_p Y \to T_p Y.$$

Restrict θ to $G \times \{p\}$; then

$$d\theta|_{(e,p)} \colon X_e \to \hat{X}_p.$$

The set of all such vectors is the tangent space to the orbit at p, which is $T_p(G \cdot p) = D_p$. Hence $d\theta_{(e,p)}$ is surjective.

Now let θ be restricted to $G \times Y$. The image of the differential at (e,p) is all of $T_p M$; hence by the Rank Theorem, there is a neighborhood of (e,p) on which (the restriction of) θ is a diffeomorphism. Shrinking if necessary, we can get a product neighborhood $X \times Y$ and a diffeomorphism ψ on $X \times Y$.

Since ψ is a diffeomorphism, it is injective. Then if $g, g' \in X$ and $g \cdot Y \cap g' \cdot Y \neq \varnothing$, we must have

$$\psi(g, Y) \cap \psi(g', Y) \neq 0 \implies g = g'.$$

Note that this statement is **not** explicitly written in the book's proof; things are much clearer once you write this down.

The proof goes by contradiction. Assume that there is no adapted neighborhood centered at p. Then we can inductively build sequences $p_i \in Y$ and $q_i = g_i \cdot p_i \in Y$ with coordinates bounded by $1/i$ for p_i, q_i in some choice of charts. By (b) from the characterization of proper actions, $\{g_i\}$ has a convergent subsequence. Pass to a subsequence $g_i \to g$. Then $g \cdot p = \lim_{i\to\infty} g_i \cdot p_i = \lim_{i\to\infty} q_i = p$. But the action is free, so $g = e$.

Now for large enough i, $(g_i, p_i) \in X \times Y$. Then $\psi(g_i, p_i) = g_i \cdot p_i \in Y$. Hence both $g \cdot Y$ and $g' \cdot Y$ intersect. The point of the properness argument was to get our subsequence that converges to e; then the tail of our subsequence is in our neighborhood. Hence $g_i = e$, whereupon $p_i = q_i$; thus they are two different points in the same orbit, which is a contradiction. □

In comparison, the rest of the chapter is easy; it is just Math 544 results + smoothness.

Proposition 24.1 (Lee, 21.12)**.** $\mathrm{Aut}_\pi E$ *acts smoothly, freely, and properly on* E.

Proposition 24.2 (Lee, 21.13)**.** *If* Γ *acts smoothly, freely, properly on* E*, then we geta smooth normal covering map.*

Homogeneous Spaces

Definition 24.3. A **homogeneous (G-)space** or G-**manifold** is a (topological or) smooth manifold with transitive (continuous or) smooth action of G.

Theorem 24.4 (Homogeneous Space Construction, 21.17)**.** *Constructing a homogeneous space from* G/H.

Proof. Given a Lie group G and a closed subgroup H, by the closed subgroup theorem H is a Lie subgroup. The right action of H on G is smooth, free, and proper. It takes a moment to check proper, but the rest is clear.

Now we change all the left actions in the quotient manifold theorem to right actions. Then we get

$$\begin{array}{ccc} G \times G & \xrightarrow{m} & G \\ \downarrow{\scriptstyle \mathrm{Id}_G \times H} & & \downarrow{\scriptstyle \pi} \\ G \times G/H & \overset{\theta}{\dashrightarrow} & G/H \end{array}$$

$\pi \circ m$ descends to the quotient; indeed, right and left actions commute with one another (that's why we had to switch our actions). Then you check that θ is a transitive left action. □

Theorem 24.5 (Homogeneous Space Characterization Theorem, 21.18)**.** *These are the only homogeneous spaces*

Proof. If M is a homogeneous space, then for any $p \in M$, G_p is a clsoed subgroup (the preimage of a single point under the orbit map).

$$\begin{array}{ccc} G & & \\ \downarrow{\scriptstyle \pi} & \searrow{\scriptstyle \theta^{(p)}} & \\ G/G_p & \overset{F}{\dashrightarrow} & M \end{array}$$

You check that $\theta^{(p)}$ is constant on the fibers of π. Conversely, π is constant on its fibers; hence F is a diffeomorphism. Check that F is equivariant with respect tothe G actions. □

The next proposition was point 3 from the outline; it is one of the most important results of this chapter, in fact!

Proposition 24.6 (Lee, 21.20)**.** *Given a set with a transitive* G*-action, we construct a homogeneous manifold structure.*

Proof. Take $\theta\colon G \times X \to X$. Assume that G_p is closed in G for some $p \in X$. Then we pass to the quotient as before, and put the smooth structure on F by making it a diffeomorphism. □

The promised easy way to prove that the Grassmanian is a manifold.

Why the flag manifolds are called flag manifolds: suppose you have a line in $\mathbb{R}^3$ contained in a plane. We picture this like a flag.

Math 546 Manifolds Notes by Avi Levy

Lecture 25: May 28

Orientation (Chapter 15)

For a vector space, we can specify an orientation by:

1. one basis (E_i)
2. an equivalence class of all bases that are conisistently ordered:

$$E_i = B_i^j \overline{E_j} \qquad \det B_i^j > 0.$$

3. a non-zero n-covector Ω

$$\Omega(E_1, \cdots, E_n) > 0; \qquad \Omega = f\, \varepsilon^1 \wedge \cdots \wedge \varepsilon^n, f > 0.$$

On a manifold M, a choice of orientation is a continuous choice of orientation on each T_pM. The proceed is:

1. Choose local continuous frames whose domains cover M and which are consistently oriented on their overlaps; call this positively oriented.
2. As a special case, pick an atlas of coordinate charts that are consistently ordered on overlaps.

Proposition 25.1 (Lee, 15.5)**.** *If M admits a consistently ordered atlas, then M is orientable. Moreover if M is oriented, then you have such an atlas.*

Definition 25.2. M is **orientable** means we can choose an orientation. M is **oriented** once we have chosen an orientation. For short, when we say **"oriented frame"** we mean **"positively oriented"**.

Proposition 25.3 (Lee, 15.6)**.** *If $\dim M \geq 1$, then a non-vanishing $\Omega \in \Omega^n(M)$ determines an orientation of M, and if M is oriented then there is a non-vanishing Ω with this property.*

Proof. Suppose we're given an Ω. Then we define a frame (E_i) to be **(positively) oriented** whenever $\Omega(E_1, \cdots, E_n) > 0$ everywhere on M. Now using the identity

$$\Omega(F_1, \cdots, F_n) = \det(B_j^i)\Omega(E_1, \cdots, E_n),$$

it follows that we have the right compatibility condition. Note that B_j^i is a matrix function depending on p, but since it varies smoothly everything is fine.

For the converse, pick an atlas of **oriented** coordinate charts (U_α, ϕ_α), and define

$$\Omega_\alpha = dx^1 \wedge \cdots \wedge dx^n$$

on U_α. Choose a partition of unity $\{\psi_\alpha\}$ subordinate to the U_α. Then set

$$\Omega = \sum \psi_\alpha \Omega_\alpha.$$

Now Ω is never the 0 n-covector, since at any point $p \in U_\alpha \cap U_\beta$ we consider

$$\Omega_\alpha(E_1, \cdots, E_n), \qquad \Omega_\beta(E_1, \cdots, E_n).$$

Then we get positive numbers, and the same result extends to any finite number of overlapping charts. Hence

$$\Omega_p(E_1, \cdots, E_n)$$

is a weighted average of positive numbers, with at least one non-zero term ψ_α. Hence $\Omega_p(E_1, \cdots, E_n) > 0$.

Note: We are technically using convexity of the positive real axis (when you average a bunch of positive real numbers you end up with a positive real). □

In the $\dim M = 0$ case, an orientation is a choice of ± 1 to each point. The reason we mention this degenerate case is because we want Stokes' Theorem to generalize the fundamental theorem of calculus.

$$\int_{[a,b]} \frac{df}{dt}\, dt = f(b) - f(a).$$

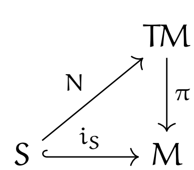
This corresponds to the induced boundary orientation on $[a, b]$:

$$\partial[a, b] = \{b\} - \{a\}.$$

Orientation of Hypersurfaces

The main application will be to the boundary of a manifold. So consider an immersed submanifold $S^n \subset M^{n+1}$ with M embedded. Using forms, we have an $(n+1)$-form, Ω, on M.

Proposition 25.4 (Lee, 15.21). *If there is a field N along S that is nowhere tangent to S, then $i_S^*(N \lrcorner \Omega)$ is nonvanishing on S, and is therefore an orientation on S.*



Recall that verticality means $T_pM = N_p + T_pS$.

Example 25.5. $M = \mathbb{R}^3$, $S = \mathbb{S}^2$, and $\Omega = dx \wedge dy \wedge dz$. It follows that $N \lrcorner \Omega$ is a 2-form that gives rise to an orientation on N.

Proposition 25.6 (Lee, 15.23)**.** *If S is a regular level set of $f \in C^\infty(M)$ and M is Riemannian, we can use $N = \operatorname{grad} f$.*

For $S = \partial M$, recall from Problem 8-4 that there is a nowhere vanishing inward/outward field along S; say you're given $\Omega = dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{H}^{n+1}$. Then on a boundary chart, you would use $\frac{\partial}{\partial x^{n+1}}$ for the inward normal and $-\frac{\partial}{\partial x^{n+1}}$ for the outward normal.

We make a convention so that Stokes' Theorem becomes more elegant. Use the outward normal, $N = -\frac{\partial}{\partial x^{n+1}}$, on the boundary, when inducing the boundary orientation. For example, consider a 3-manifold with boundary like $\mathbb{H}^3$. Then

$$-\frac{\partial}{\partial x^3} \lrcorner dx^1 \wedge dx^2 \wedge dx^3 = -dx^1 \wedge dx^2.$$

The minus signs come from the fact that we interior multiply on the first slot, but work with inward/outward on the last slot.

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Notes by Avi Levy

Lecture 26: May 30

The Orientation Covering

For $p \in M$, there are 2 orientations for T_pM. We have a covering map $\hat{\pi}\colon \hat{M} \to M$, which identifies things of opposite orientation. For instance, consider when M is a hypersurface in $\mathbb{R}^{n+1}$. Then in problem 15-15, we work with the unit normal N at p given by

$$N \lrcorner\, dx^1 \wedge \cdots \wedge dx^{n+1}.$$

This gives an orientation for T_pM.

Theorem 26.1 (Lee, 15.40)**.** *For $\hat{\pi}\colon \hat{M} \to M$, $\hat{M}$ can be given the structure of a smooth* **oriented** *manifold.*

(a) $\hat{\pi}$ is a generalized covering map

(b) For connected open $U \subset M$, we have that U is evenly covered precisely when it is orientable.

(c) If U is evenly covered, the 2 orientations of U are σ^* of an orientation on $\hat{M}$, where σ is a section of $\hat{M} \to M$.

Example applications: If M is not orientable, that tells us something about $\hat{M}$. For instance, see 16.48 (densities) and 17.34 (top cohomology groups).

Proof sketch. (a) We form the pairs $(p, \mathcal{O}_p)$, and make it the basis for a topology on $\hat{M}$. This involves checking properties on the intersections. Since U is evenly covered, $\hat{\pi}$ is a generalized covering map. Observe that $\hat{\pi}$ restricted to any component of $\hat{M}$ is an ordinary covering map. We use the pullback smooth structure (an earlier result), and obtain that the orientation at $(p, \mathcal{O}_p)$ is $\hat{\pi}^*(\mathcal{O}_p)$, and likewse the orientation for $(p, -\mathcal{O}_p)$ is $\hat{\pi}^*(-\mathcal{O}_p)$.

(b) We already did one direction:

- $\Rightarrow$ When U is evenly covered, then $\pi^{-1}(U)$ has 2 shoots, each diffeomorphic to U. Each sheet is in $\hat{M}$ which is oriented, thus U can be oriented.
- $\Leftarrow$ See part (a)

(c) While doing part (a), we constructed such an orientation.

□

Theorem 26.2 (Lee, 15.41)**.** *M is orientable precisely when $\hat{M}$ has 2 components, and M is non-orientable precisely when $\hat{M}$ is connected and $\hat{\pi}$ is a double cover.*

Proof. If M is oriented, then by part (b) it is evenly covered. Hence it is double covered. Then we can pick $(p, \mathcal{O}_p)$ and $(p, -\mathcal{O}_p)$ to get our two components.

Conversely, if $\widehat{M}$ is disconnected then $\widehat{\pi}$ is a diffeomorphism when restricted to a component (by the Global Rank Theorem). Then you can invert the restriction of π, to obtain a global section. Use this to pull back the orientation of $\widehat{M}$; it follows that M is oriented. □

Now there's a proposition about factoring through the orientation cover, which gives you uniqueness. Next there's a nice result about subgroups of index 2 in the fundamental group.

Integration

We start on $\mathbb{R}^n$. There are two different types of integration going on here. First, we have $\int_E f\, dx^1 \cdots dx^n$ on Euclidean, when we can use results like Fubini's Theorem to interchange the order. For instance, we can integrate a curve with respect toarclength and do its Euclidean integral.

Then, there's integration with orientation. Think of line integrals or contour integration, when the integral changes sign if you switch the orientation. Integration of forms falls into the second category.

Definition 26.3. Define

$$\int_E f\, dx^1 \wedge \cdots \wedge dx^n = \int_E f\, dx^1 \cdots dx^n.$$

The order matters on the first version, where we are using the standard orientation on $\mathbb{R}^n$. To compute in a different order

$$\int_E g\, dx^2 \wedge dx^1 \wedge \cdots \wedge dx^n = -\int_E g\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Example on $\mathbb{R}^2$: we can look at $\int_E g\, dy \wedge dx = -\int_E g\, dx\, dy$. Thus integration of forms takes into account orientation information that was previously absent on $\mathbb{R}^n$. Now consider a hypersurface D contained in the xz-plane as a subspace of $\mathbb{R}^3$. Now to compute ϵ_D $f(x, z)\, dx \wedge dz$, we need to know "which way is up"; for instance, if $\frac{\partial}{\partial y}$ is your "positive" normal, then the orientation for D is

$$\frac{\partial}{\partial y} \lrcorner\, dx \wedge dy \wedge dz = -dx \wedge dz.$$

This is inspired by physical interpretations, for instance if you have a fluid flowing through a region.

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Lecture 27: June 2

A few technicalities: extending the theory to improper integrals. We're not going to do this; everything is on compact domains. In particular, you can interchange the order of integration, because convergence is uniform.

Consider an n-form ω on $\mathbb{R}^n$ with compact support. The support could have a boundary with positive measure, but the theory has to allow for that. So we want $\operatorname{supp}\omega$ contained in a domain of integration.

Definition 27.1. A **domain of integration** is a subset of $\mathbb{R}^n$ that is bounded, whose boudnary has measure 0.

Recall that from Chapter 6, measure is well-defined on manifolds, and we extend our definition to there. Last time, we defined

$$\int_{D\subset\mathbb{R}^n}\omega=\int_D f\,dx^1\wedge\cdots\wedge dx^n=\int_D f\,dx^1\cdots dx^n.$$

To extend this to manifolds, we need an orientation.

Now consider ω on an oriented manifold M. Take an oriented chart (U,ϕ) such that $\operatorname{supp}\omega\cap U$ is compact. Then define

$$\int_U\omega=\int_{\phi(U)}\phi^{-1\,*}\omega.$$

To generalize, we use partitions of unity (check that the result is well-defined). This approach is good for some theoretical arguments, but it is not useful for actual computations (since bump functions are hard to work with).

Instead, we use parametrizations to compute integrals:

Proposition 27.2 (Lee, 16.8)**.** *Consider a finite collection D_i of open domains of integration in $\mathbb{R}^n$. Suppose that*

(i) Each F_i is orientation preserving and extends smoothly to $\overline{D_i}\to M$

(ii) $\{W_i\}$ is a partition

(iii) $\operatorname{supp}\omega\subset\cup_i\overline{W_i}$

Then we have

$$\int_M\omega=\sum_i\int_{D_i}F_i^*\omega.$$

For an example, see [Lee, 16.9]. We compute $\int_{\mathbb{S}^2} \omega$. Take F to be the inverse of spherical coordinates. Then F is a diffeomorphism for $\theta \in (0, 2\pi)$ and $\phi \in (0, \pi)$. Then D_1 is an open rectangle, and

$$W_1 = \mathbb{S}^2 \setminus \{\text{a prime meridian}\}.$$

The integral you get here is the surface area (the standard Reimannian volume form in this case).

Stokes' Theorem

The statement written in chalk on the ceiling of the classroom was

$$\int_M d\omega = \int_{\partial M} \omega.$$

The original statement holds for M an n-dimensional oriented manifold (possibly with boundary), and where ∂M has the induced orientation. Also ω is a compactly supported $(n-1)$-form on M. The notation $\int_{\partial M} \omega$ is interpeted to mean $\int_{\partial M} i^*\omega$, where $i \colon \partial M \to M$ is the inclusion map. While we rarely write in this detail, it is sometimes helpful while solving homework problems.

Let's see what happens when ω is not compactly supported.

1. Consider a compactly supported ω on a manifold with boundary M. Choose M such that $\partial M \neq \varnothing$, and choose ω such that $\int_{\partial M} \omega \neq 0$. Now consider the auxiliary manifold $N = \operatorname{Int} M$. Since N is open, it follows by invariance of boundary that $\partial N = \varnothing$. In particular, $\int_{\partial N} \omega = 0$. However,

$$\begin{aligned} \int_N d\omega &= \int_M d\omega \\ &= \int_{\partial M} \omega \\ &\neq 0. \end{aligned}$$

   Therefore Stokes' Theorem does not hold for the pair M, ω.

2. More concretely, consider

$$\omega = \frac{x\, dy - y\, dx}{x^2 + y^2} ='' d\theta''.$$

   We see that $d\omega = 0$. Take $M = \mathbb{R}^2 \setminus B(0, 1)$. Then $\int_M d\omega = 0$, whereas $\int_{\partial M} \omega = -2\pi$.

There are many useful corollaries of Stokes' Theorem.

Corollary 27.3. *Let M be a smooth compact oriented manifold, and ω an $(n-1)$-form.*

16.13 If $\alpha = d\omega$ *and* $\partial M = \varnothing$*, then*

$$\int_M \alpha = \int_m d\omega = 0.$$

16.14 If ω *is closed when* $\int_{\partial M} \omega = 0$.

16.15 Suppose ω *is a closed* k*-form on* M*,* S *is a compact oriented* k*-dimensional submanifold without boundary, and* $\int_S \omega \neq 0$*. Then* ω *is not exact on* M*, and* $S \neq \partial N$ *for any compact oriented submanifold with boundary* N*.*

To illustrate 16.15, take $M = \mathbb{R}^2 \setminus \{(0,0)\}$, $\omega ='' d\theta''$ as before, and $S = \mathbb{S}^1$. Then $\int_S \omega = 2\pi \neq 0$. Hence by 16.15, if we keep ω as defined on points of $\mathbb{S}^1$ and modify it smoothly however we like off of $\mathbb{S}^1$, we can't get it to be exact. Also, there is no compact $N \subset M$ with $\partial N = \mathbb{S}^1$.

Just using Stokes' Theorem, we can prove a result about de Rham cohomology from Chapter 17.

Definition 27.4. The **de Rham cohomology groups** are given by

$$H^k_{\text{de Rham}}(M) = \frac{\ker d\colon \Omega^k \to \Omega^{k+1}}{\operatorname{Im} d\colon \Omega^{k-1} \to \Omega^k}.$$

Corollary 27.5 (Part of 17.30)**.** *For a compact oriented* n*-manifold without boundary* M*, we have*

$$H^n_{\textit{de Rham}}(M) \neq \{0\}.$$

Proof. There is an orientation form Ω; we claim that $\int_M \Omega > 0$, since it is positive on each oriented chart. Hence by 16.13, Ω is not exact. On the other hand, $d\Omega = 0$ since Ω is a top-dimension form. ☐

Proof of Stokes' Theorem. By partitions of unity, it suffices to prove the result for when $\operatorname{supp}\omega$ is contained in a single chart. For simplicity, put a rectangular domain of integration around $\operatorname{supp}\omega$ (technically, we realy mean $\phi^{-1\,*}\omega$.

We start by working in an interior chart. By linearity, it suffices to consider

$$\omega = f\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$
$$d\omega = \pm \frac{\partial f}{\partial x^i}\, dx^1 \wedge \cdots \wedge dx^n.$$

Integrate with respect tox^i to get f evaluated at 2 points on ∂A, where it vanishes. Since $\partial A = \varnothing$, both sides agree.

Now consider a boundary chart. The same thing happens unless $i = n$, so we restrict attention to this case. For notational convenience, we work with $n = 3$. Then the Stokes' orientation on the xy-plane is given by

$$-\frac{\partial}{\partial z} \lrcorner\, dx \wedge dy \wedge dz = -dx \wedge dy.$$

As before, take $\omega = f\ dx \wedge dy$. Then

$$d\omega = \frac{\partial f}{\partial z}\ dx \wedge dy \wedge dz.$$

Consequently we compute that

$$\begin{aligned}
\int_A d\omega &= \int_A \frac{\partial f}{\partial z}\ dx\ dy\ dz \\
&= \int_{B = A \cap \partial \mathbb{H}^n} \left(\int_0^z \frac{\partial f}{\partial z}\ dz \right)\ dx\ dy \\
&= \int_B f(x,y,z) - f(x,y,0)\ dx\ dy \\
&= -\int_B f(x,y,0)\ dx\ dy \\
&= \int_{\tilde{B}} \omega.
\end{aligned}$$

Here $\tilde{B}$ is B with the Stokes' orientation. □

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Lecture 28: June 4

Divergence of a vector field

Given a volume form Ω (not necessarily a Riemannian metric), we can define the divergence as follows:

$$\operatorname{div}(X)\Omega = \mathcal{L}_X\Omega.$$

This is equivalent to the usual definition, by Cartan's Formula. Indeed, $d\Omega$ vanishes since it is an $(n+1)$-form on an n manifold.

If $X \neq 0$, then working in flow coordinates we have $X = \frac{\partial}{\partial x^1}$, so that $\mathcal{L}_X dx^i = 0$. Writing $\Omega = f\, dx^1 \wedge \cdots \wedge dx^n$, we obtain

$$\mathcal{L}_X\Omega = \frac{\partial f}{\partial x^1}\, dx^1 \wedge \cdots \wedge dx^n = \left(\frac{1}{f}\frac{\partial f}{\partial x^1}\right)\Omega.$$

Hence we thikn of $\operatorname{div}(X)$ like $\left(\frac{1}{f}\frac{\partial f}{\partial x^1}\right)$.

Proposition 28.1 (Lee, 16.33)**.** *If D is a compact regular domain (codimension 0 submanifold with boundary), then*

$$\begin{aligned}\frac{d}{dt}\mathit{vol}\theta_t(D) &= \frac{d}{dt}\int_{\theta_t(D)}\Omega \\ &= \frac{d}{dt}\int_D \theta_t^*\Omega \\ &= \int_D \frac{d}{dt}\theta_t^*\Omega.\end{aligned}$$

The last line follows since D is compact and all the functions are continuous (so the convergence is uniform). Hence

$$\begin{aligned}&= \int_D \theta_t^*\mathcal{L}_X\Omega \\ &= \int_{\theta_t(D)} \mathcal{L}_X\Omega \\ &= \int_{\theta_t(D)} (\operatorname{div} X)\Omega.\end{aligned}$$

If $\Omega = dV_g$ (the Riemannian volume form), we get the classic divergence theorem. Note that if M is compact without boundary, then dV_g is not actually an exact form, by Stokes' Theorem - the notation is misleading. This is Proposition 16.32.

Proof. If X is compactly supported, then

$$\begin{aligned} \int_M (\operatorname{div} X)\, dV_g &= \int_M d(i_X\, dV_g) \\ &= \int_{\partial M} i_X dV_g|_{\partial M} \\ &= \int_{\partial M} i_S^*(i_X\, dV_g) \\ \text{Lemma 16.30} \qquad &= \int_{\partial M} \langle X, N\rangle dV_{\tilde{g}}. \end{aligned}$$

Here N is a normal field along ∂M, and $\tilde{g}$ is the induced Riemannian metric on ∂M. □

Lemma 28.2 (Lee, 16.30)**.** *We have the identity*

$$i_X\, dV_g|_{\partial M} = \langle X, N\rangle i_N\, dV_g,$$

and the same result holds on any hypersurface.

Proof. It suffices to show that this holds pointwise for $p \in \partial M$. Pick an orthonormal basis $E_1, \cdots, E_{n-1}$ for $T_p\partial M$. If $\langle X, N\rangle = 0$, then both sides vanish. Indeed, the left side acts on $(X, Y_1, \cdots, Y_{n-1})$ with $Y_i \in T_p\partial M$, which will be linearly dependent.

By linearity, it suffices to consider $X = aN$ (since the tangential components vanish by the previous paragraph). We evaluate both sides on $E_1, \cdots, E_{n-1}$ to obtain

$$dV_g(aN, E_1, \cdots, E_{n-1}) = a.$$

□

The message here is that when working with top degree forms, just apply them to a basis and you'll know everything.

Notice that the divergence has an interpretation in terms of the Hodge $*$ operator $*\colon \Omega^p \to \Omega^{n-p}$.

$$\operatorname{div} X = * d * X^\flat \qquad \text{see 16-18, 16-21 if interested.}$$

A brief word on densities: You can embed a Mobius band/strip in $\mathbb{R}^3$. As a hypersurface, area makes sense. For instance, $\int_M f\, dA$ makes physical sense.

de Rham Cohomology

We make the definition

$$H^p_{dR}(M) = \frac{\ker d\colon \Omega^p(M) \to \Omega^{n+1}(M)}{d(\Omega^{n-1}(M))}.$$

Two questions; why do we call this a group, and why do we put the "co" in front of homology? The answer is that it fits into a larger picture, and they are maps from a homology into $\mathbb{R}$ (which is a group in general).

The very rough answer is that the corresponding is defined on compact oriented submanifolds with boundary. Then Stokes' Theorem says that

$$\int_S d\omega = \int_{\partial S} \omega.$$

Therefore the exterior derivative d is "dual" to the boundary operator ∂. To make this precise, use singular homology. So we look at maps from a p-simplex into M. We have an operator given by integration over simplices, and in general we can have an operator that assigns a quantity to a simplex. The output can live in any group, in general. You can extend by scalars to go from the integer version to the full version. The story is in Chapter 18.

There are some facts to know about H^*_{dR}, even though we aren't going over their proofs.

1. H^*_{dR} are homotopy invariants. Thus we can compute the de Rham cohomology of contractible spaces very easily.

2. The local version of this is the Poincaré Lemma; on star-shaped sets, the constant functions are the only forms that are exact but not closed.

3. Applying Mayer-Vietoris, we compute the cohomologies for S^n and $\mathbb{R}^{n+1} \setminus \{pt\}$. It's $\mathbb{R}$ in dimensions 0 and n, and 0 otherwise.

4. For connected compact n-manifolds, we have $H^n(M) = \mathbb{R}$ with isomorphism

$$\int_M : [\omega] \mapsto \int_M \omega.$$

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Lecture 29: June 6

Possible topics for the optional lecture on Monday: Proof that

$$[\omega] \to \int_M \omega$$

is an isomorphism from $H^n_{cpt}(M^n)$ to $\mathbb{R}$.

Proof of Mayer-Vietoris.

There is a suggested problem from Chapter 18; think of a p-cycle as a map

$$\sigma\colon \mathbb{S}^p \to M.$$

Claim 29.1 (18-1)**.** *The geometric interpretation of homology: a closed* p*-form is exact whenever*

$$\int_{\sigma(\mathbb{S}^n)} \omega = 0$$

for all cycles σ*.*

Note that if σ extends to a map from $\mathbb{B}^{n+1} \to M$, then by Stokes' Theorem this is trivially satisfied.

For the rest of today, we omit the "de Rham" subscript from the cohomology groups. Note that $H^0(M)$ is the set of locally constant functions, which is a vector space under addition. In particular,

$$H^0(M) \simeq \mathbb{R}^k,$$

where k is the number of components.

Homotopy invariance via the homotopy operator. The proof generalized the Poincaré Lemma for covector fields on a starshaped domain $U \subset rr^n$. Given a covector field ω, we wanted an f such that $df = \omega$. So we defined f as the integral of ω along a radial line; in other words, integrating along the path of a homotopy.

Think of this field like $\frac{d}{dt}$. Now let S be the analogue of this field for any homotopy. On $U \times I$, the field S pushes forward to a field that is tangent to the lines of the homotopy. Hence we define

$$f = \int_0^1 i_S^* \omega \, dt.$$

Consider $i_0\colon M \to M \times I$, and $i_1\colon M \to M \times I$. Later we'll consider maps $F, G\colon M \times I \to N$. If we identify M with $M \times \{t\}$, we can write the integral

$$h(w) = \int_0^1 i_t^*(S \lrcorner \omega) \, dt \in \Omega^{p-1}(M),$$

once we're given a p-form ω on $M \times I$. If $S = \frac{d}{dt}$, then we think of $S \lrcorner \omega_{(q,t)} \in \Lambda^{p-1}(T_q^*M)$. We compute

$$\begin{aligned} d(h(\omega)) &= \int_0^1 i_t^* d(i_S\omega)\, dt \\ \implies d(h(\omega)) + h(d\omega) &= \int_0^1 i_t^*[(di_S\omega) + i_S d\omega)]\, dt \\ &= \int_0^1 i_t^* \mathcal{L}_S \omega\, dt \\ &= i_1^*\omega - i_0^*\omega. \end{aligned}$$

Thus if ω is closed, we have

$$d(h(\omega)) = i_1^*\omega - i_0^*\omega.$$

Why is this useful? If we have a p-form ω on N, then we have $F \simeq G\colon M \to N$, which means $dh(H^*\omega) = G^*\omega - F^*\omega$. Hence F, G have the same cohomology class.

Some immediate results:

1. $H^1(M) \hookrightarrow \mathrm{Hom}(\pi_1(M), \mathbb{R})$ The map takes a closed 1-form and an element $\sigma\colon S^1 \to M$ to their integral:

   $$\omega \mapsto \int_{\sigma(S^1)} \omega.$$

   For the full isomorphism, you need to abelianize $\pi_1(M)$ and extend it by scalars ($\mathbb{R}$).

2. $H^p(\mathbb{R}^n) = \delta_{0,p}\mathbb{R}$

3. $H_c^p(\mathbb{R}^n) = \delta_{n,p}\mathbb{R}$ To understand why, first consider $H_c^0(\mathbb{R}^n)$. It vanishes, since the only compactly supported constant functions is the 0 function. Now consider $p = 1$; then if $\omega = \varphi\, dx$ where φ has compact support, it follows that $\int_{\mathbb{R}} \omega = 0$. Hence if $\omega_1 - \omega_2$ is exact, then $\int_{\mathbb{R}} \omega_1 = \int_{\mathbb{R}} \omega_2$. Hence

   $$[\omega] \mapsto \int_{\mathbb{R}} \omega$$

   is an isomorphism.

This extends to all smooth connected oriented manifolds, with $H_c^n(M) \simeq \mathbb{R}$ via the integration map above.

For compact smooth connected oriented manifolds M, we have $H^n(M) \simeq \mathbb{R}$.

Theorem 29.2 (Mayer-Vietoris)**.** *Let $M = U \cup V$, where U, V are open. Then we have an l.e.s.*

$$\begin{array}{ccccc} H^p(M) & \longrightarrow & H^p(U) \oplus H^p(V) & \xrightarrow{i^* - j^*} & H^p(U \cap V) \\ & \swarrow & & & \\ H^{p+1}(M) & \longrightarrow & \cdots & & \end{array}$$

Let's apply this to $H^p(\mathbb{S}^n)$. Then $H^0(\mathbb{S}^n) = \mathbb{R}$, $H^1(\mathbb{S}^1) = \mathbb{R}$, and $H^1(\mathbb{S}^2) = 0$. Take charts U, V that miss only the north, south poles. If you continue inductively in the sequence, we get enough zeros to force the computation of the groups.