Math 309 C Winter 2015 Final
March 16, 2015

Name:____

Student ID Number:____________________________________

1	15	
2	15	
3	15	
4	15	
5	10	
Total	70	

- You have until 4:20 pm to complete the exam. There are five problems. Read all of the problems carefully before starting to work on them.
- **Turn off and put away cell phones.** You are allowed to use a scientific calculator as well as an 8.5×11 sheet of handwritten notes (you may use both sides). **No graphing calculators are allowed**
- **Show all of your work!** You will receive no credit for a problem where you just provide an answer and do not show your work. If you need more room, use the backs of pages. If you do so, please make a note for the grader (for example an arrow and a comment along the lines of "continued on back").
- You can use the back of the first and second page as the scratch paper.
- All solutions should be given in terms of **real functions (no imaginary units)**.
- If you have any questions, please raise your hand.
- Good luck!

Some formulas:

$$\int x\sin(Ax)dx = -\frac{x}{A}\cos(Ax) + \frac{1}{A^2}\sin(Ax) + c$$
$$\int x\cos(Ax)dx = \frac{x}{A}\sin(Ax) + \frac{1}{A^2}\cos(Ax) + c$$
$$\cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\sin^2 x = \frac{1-\cos(2x)}{2}$$
$$\sin(2x) = 2\sin x\cos x$$

Fourier series. Each of f_a, f_b, f_c and f_d is given on the interval $(-L, L)$ and is extended to be $2L$-periodic.

$$f_a(x) = \begin{cases} -1, & \text{for } -L < x < 0 \\ 1, & \text{for } 0 < x < L \end{cases} \qquad FS = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\frac{(2k-1)\pi x}{L}.$$

$$f_b(x) = x \text{ for } -L < x < L \qquad FS = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi x}{L}.$$

$$f_c(x) = |x| \text{ for } -L < x < L \qquad FS = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi x}{L}.$$

$$f_d(x) = \begin{cases} 0, & \text{for } -L < x < -L/2 \\ 1, & \text{for } -L/2 < x < L/2 \\ 0, & \text{for } L/2 < x < L. \end{cases} \qquad FS = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\cos\frac{(2k-1)\pi x}{L}$$

Here are two functions given on $(0, L)$, and their cosine and sine series with period $2L$.

$$f_e(x) = \begin{cases} 0, & \text{for } 0 < x < L/4 \\ 1, & \text{for } L/4 < x < 3L/4 \\ 0, & \text{for } 3L/4 < x < L. \end{cases} \qquad \text{cosine series} = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}\cos\frac{(4k-2)\pi x}{L};$$

$$\text{sine series} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi/4) - \cos(3n\pi/4)}{n}\sin\frac{n\pi x}{L}$$

$$f_f(x) = \begin{cases} x, & \text{for } 0 < x < L/2 \\ L - x, & \text{for } L/2 < x < L \end{cases} \qquad \text{cosine series} = \frac{L}{4} - \frac{2L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(4k-2)\pi x}{L}$$

$$\text{sine series} = \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}\sin\frac{(2k-1)\pi x}{L}$$

1. (15 points) Find the general solution to

$$x' = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} x + \begin{pmatrix} 3e^t \\ e^{-t} \end{pmatrix}.$$

You can use diagonalization, undetermined coefficients, or variation of parameters. CIRCLE the method you'll use.

$A - rI = \begin{pmatrix} 3-r & -2 \\ 5 & -4-r \end{pmatrix}$, $\det(A - rI) = r^2 + r - 2 = 0$, so $r_1 = -2$ and $r_2 = 1$.

When $r_1 = -2$, we have

$$\begin{pmatrix} 5 & -2 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \xi^{(1)} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \Rightarrow x^{(1)} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{-2t}.$$

When $r_2 = 1$, we have

$$\begin{pmatrix} 2 & -2 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow x^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t}.$$

1). Diagonalization.

Use $T = \begin{pmatrix} 2 & 1 \\ 5 & 1 \end{pmatrix}$ and $x = Ty$, thus, with $T^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -1 \\ -5 & 2 \end{pmatrix}$, y satisfies the system

$$y' = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} y + T^{-1}g = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} y + \begin{pmatrix} -e^t + \frac{1}{3}e^{-t} \\ 5e^t - \frac{2}{3}e^{-t} \end{pmatrix}.$$

Thus,

$$y_1' = -2y_1 - e^t + \frac{1}{3}e^{-t}$$
$$y_2' = y_2 + 5e^t - \frac{2}{3}e^{-t}.$$

We can solve y_1 and y_2

$$y_1 = e^{-2t}\int e^{2s}(-e^s + \frac{1}{3}e^{-s})ds + c_1e^{-2t} = -\frac{1}{3}e^t + \frac{1}{3}e^{-t} + c_1e^{-2t};$$
$$y_2 = e^t\int e^{-s}(5e^s - \frac{2}{3}e^{-s})ds + c_2e^t = 5te^t + \frac{1}{3}e^{-t} + c_2e^t.$$

The solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Ty = \begin{pmatrix} 2 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{3}e^t + \frac{1}{3}e^{-t} + c_1e^{-2t} \\ 5te^t + \frac{1}{3}e^{-t} + c_2e^t \end{pmatrix}$$

2). Undetermined coefficients.

$g(t) = e^t \begin{pmatrix} 3 \\ 0 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, let the particular solution $v(t) = \vec{a}te^t + \vec{b}e^t + \vec{c}e^{-t}$, then

$$\vec{a}te^t + \vec{a}e^t + \vec{b}e^t - \vec{c}e^{-t} = A\vec{a}te^t + A\vec{b}e^t + A\vec{c}e^{-t} + e^t \begin{pmatrix} 3 \\ 0 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Comparing coefficients,

$$\vec{a} = A\vec{a}$$
$$\vec{a} + \vec{b} = A\vec{b} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$
$$-\vec{c} = A\vec{c} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$\vec{a}$ is an eigenvector, so $\vec{a} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}$, then from the second equation,

$$(A - I)\vec{b} = \begin{pmatrix} \alpha - 3 \\ \alpha \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -2 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha - 3 \\ \alpha \end{pmatrix} \Rightarrow 5(\alpha - 3) = 2\alpha.$$

So $\alpha = 5$, $\vec{a} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. From the third equation, $\vec{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
Thus, the general solution is

$$x(t) = c_1 \begin{pmatrix} 2 \\ 5 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 5 \\ 5 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}.$$

3). Variation of parameters.
A fundamental matrix is

$$\Psi(t) = \begin{pmatrix} 2e^{-2t} & e^t \\ 5e^{-2t} & e^t \end{pmatrix} \text{ and } \Psi^{-1}(t) = \begin{pmatrix} -\frac{e^{2t}}{3} & \frac{e^{2t}}{3} \\ \frac{5e^{-t}}{3} & -\frac{2e^{-t}}{3} \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \Psi^{-1}(t)g(t) = \begin{pmatrix} -\frac{e^{2t}}{3} & \frac{e^{2t}}{3} \\ \frac{5e^{-t}}{3} & -\frac{2e^{-t}}{3} \end{pmatrix} \begin{pmatrix} 3e^t \\ e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{3t} + \frac{1}{3}e^t \\ 5 - \frac{2}{3}e^{-2t} \end{pmatrix}.$$

So we have

$$u(t) = \begin{pmatrix} -\frac{1}{3}e^{3t} + \frac{1}{3}e^{t} + c_1 \\ 5t + \frac{1}{3}e^{-2t} + c_2 \end{pmatrix};$$

$$x(t) = \Psi(t)u(t) = \begin{pmatrix} 2e^{-2t} & e^{t} \\ 5e^{-2t} & e^{t} \end{pmatrix} \begin{pmatrix} -\frac{1}{3}e^{3t} + \frac{1}{3}e^{t} + c_1 \\ 5t + \frac{1}{3}e^{-2t} + c_2 \end{pmatrix}.$$

2. Consider the homogeneous system with parameter α:

$$x' = \begin{pmatrix} -1 & 2 & 4 \\ 0 & \alpha & \alpha+5 \\ 0 & \alpha-5 & \alpha \end{pmatrix} x.$$

(a) (6 points) Find the general solution when $\alpha = 5$.

For $\alpha = 5$ we have $A = \begin{pmatrix} -1 & 2 & 4 \\ 0 & 5 & 10 \\ 0 & 0 & 5 \end{pmatrix}$ and $A - rI = \begin{pmatrix} -1-r & 2 & 4 \\ 0 & 5-r & 10 \\ 0 & 0 & 5-r \end{pmatrix}$ and $\det(A - rI) = (-1-r)(5-r)^2 = 0$, the eigenvalues are $r_1 = -1$ and $r_{2,3} = 5$.
When $r_1 = -1$ we have

$$\begin{pmatrix} 0 & 2 & 4 \\ 0 & 6 & 10 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t}.$$

When $r_{2,3} = 5$ we have

$$\begin{pmatrix} -6 & 2 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \xi^{(2)} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \Rightarrow x^{(2)} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} e^{5t}.$$

To find the generalized eigenvector,

$$\begin{pmatrix} -6 & 2 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \Rightarrow \eta = \begin{pmatrix} 0 \\ -1/10 \\ 3/10 \end{pmatrix} \Rightarrow x^{(3)} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} te^{5t} + \begin{pmatrix} 0 \\ -1/10 \\ 3/10 \end{pmatrix} e^{5t}.$$

The genral solution is

$$x(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} e^{5t} + c_3 \left[\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} te^{5t} + \begin{pmatrix} 0 \\ -1/10 \\ 3/10 \end{pmatrix} e^{5t} \right]$$

(b) (6 points) Find the general solution when $\alpha = 3$.

For $\alpha = 3$ we have $A = \begin{pmatrix} -1 & 2 & 4 \\ 0 & 3 & 8 \\ 0 & -2 & 3 \end{pmatrix}$ and $A - rI = \begin{pmatrix} -1-r & 2 & 4 \\ 0 & 3-r & 8 \\ 0 & -2 & 3-r \end{pmatrix}$ and $\det(A - rI) = (-1-r)[(r-3)^2 + 16] = 0$, the eigenvalues are $r_1 = -1$ and $r_{2,3} = 3 \pm 4i$. When $r_1 = -1$ we have

$$\begin{pmatrix} 0 & 2 & 4 \\ 0 & 4 & 8 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t}.$$

For $r_2 = 3 + 4i$ we have

$$\begin{pmatrix} -4-4i & 2 & 4 \\ 0 & -4i & 8 \\ 0 & -2 & -4i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \xi^{(2)} = \begin{pmatrix} 1 \\ 2 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

With $\lambda = 3$, $\mu = 4$, $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we have solutions

$$x^{(2)} = e^{3t}\left[\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cos(4t) - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin(4t) \right]; x^{(3)} = e^{3t}\left[\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \sin(4t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos(4t) \right]$$

The general solution is

$$x(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t} + c_2 e^{3t}\left[\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cos(4t) - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \sin(4t) \right] + c_3 e^{3t}\left[\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \sin(4t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos(4t) \right]$$

(c) (3 points) Consider $-\infty < \alpha < \infty$, for what values of α does the general solution converge to $\mathbf{0}$ as $t \to \infty$?

$A - rI = \begin{pmatrix} -1-r & 2 & 4 \\ 0 & \alpha - r & \alpha + 5 \\ 0 & \alpha - 5 & \alpha - r \end{pmatrix}$ and $\det(-1-r)[(r-\alpha)^2 - \alpha^2 + 25] = (-1-r)(r^2 - 2\alpha r + 25) = 0$, $r_1 = -1$ and $r_{2,3} = \alpha \pm \sqrt{\alpha^2 - 25}$.
If the general solution coverges to 0 as $t \to \infty$, then two eigenvalues are both negative or the real part of complex eigenvalues is negative. So $\alpha < 0$.

3. Consider an elastic string of length 20 meters with $a^2 = 9$. One end $x = 0$ is held fixed at 0, while the other end $x = 20$ is held fixed at 10. The string is set in motion with no initial velocity from the initial position $3x$ for $0 < x < 20$.

(a) (2 points) Write the differential equation, boundary conditions, and initial conditions that the displacement $u(x,t)$ should satisfy.

$$\begin{cases} 9u_{xx} = u_{tt}; \\ u(0,t) = 0, u(20,t) = 10, \ t > 0; \\ u(x,0) = 3x, u_t(x,0) = 0, \ 0 < x < 20. \end{cases}$$

(b) (3 points) Note that the boundary conditions are nonhomogeneous, so we want to find the steady-state displacement $v(x) = \lim_{t\to\infty} u(x,t)$. Solve for $v(x)$.

With $v(x) = \lim_{t\to\infty} u(x,t)$, $v''(x) = 0$, so $v(x) = k_1 x + k_2$. From the boundary conditions, $v(0) = 0$ and $v(20) = 10$, we get

$$v(x) = \frac{x}{2}.$$

(c) (3 points) We write the solution as $u(x,t) = v(x) + w(x,t)$, find the differential equation, initial and boundary conditions that $w(x,t)$ should satisfy .

From the assumption, $w(x,t) = u(x,t) - v(x)$, w satisfies

$$\begin{cases} 9w_{xx} = w_{tt}; \\ w(0,t) = 0, w(20,t) = 0, \ t > 0; \\ w(x,0) = 3x - \frac{x}{2} = \frac{5}{2}x, w_t(x,0) = 0, \ 0 < x < 20. \end{cases}$$

(d) (7 points) Find the displacement $u(x,t)$. You are **NOT** required to start with separation of variables.

Similar as the wave equation, with $L = 20$ and $a = 3$,

$$w(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{20} \cos\frac{3n\pi t}{20}.$$

With $t = 0$,

$$w(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{20} = \frac{5}{2}x = \frac{5}{2} \text{ sine series of } f_b = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi x}{20}.$$

Thus, $c_n = \frac{100(-1)^{n-1}}{n\pi}$ and the solution is

$$u(x,t) = \frac{x}{2} + \sum_{n=1}^{\infty}\frac{100(-1)^{n-1}}{n\pi}\sin\frac{n\pi x}{20}\cos\frac{3n\pi t}{20}$$

4. Consider the partial differential equation

$$u_{xx} = u_t + \gamma u, \qquad 0 < x < L,\ t > 0$$

with boundary conditions and initial condition

$$u_x(0,t) = 0,\ u_x(L,t) = 0, \qquad t > 0,$$
$$u(x,0) = f(x), \qquad 0 < x < L.$$

(a) (3 points) Use separation of variables, write down the differential equations that X and T must satisfy.
Use $u(x,t) = X(x)T(t)$ and plug into the differential equation,

$$X''T = XT' + \gamma XT \Rightarrow \frac{X''}{X} = \frac{T'}{T} + \gamma = -\lambda,$$

$$\begin{cases} X'' + \lambda X = 0 \\ T' + (\lambda + \gamma)T = 0 \end{cases}$$

(b) (12 points) Find the solution $u(x,t)$.
With zero conditions $X'(0) = 0, X'(L) = 0$, the eigenvalues and eigenfunctions are

$$\lambda_0 = 0, X_0 = 1; \lambda_n = \frac{n^2\pi^2}{L^2}, X_n = \cos\frac{n\pi x}{L}.$$

For $\lambda_0 = 0$,

$$T' + \gamma T = 0 \Rightarrow T = e^{-\gamma t};$$

For $\lambda_n = n^2\pi^2/L^2$,

$$T' + (\frac{n^2\pi^2}{L^2} + \gamma)T = 0 \Rightarrow T = e^{-(\gamma + n^2\pi^2/L^2)t}.$$

The solution is

$$u(x,t) = \frac{c_0}{2}e^{-\gamma t} + \sum_{n=1}^{\infty} c_n e^{-(\gamma+n^2\pi^2/L^2)t}\cos\frac{n\pi x}{L}.$$

When $t = 0$,

$$u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{L} = f(x),$$

we have

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx,\ n = 0, 1, 2, \ldots.$$

5. (10 points) Consider the Laplace's equation

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 10,\ 0 < y < 5 \\ u(x,0) = 0, u(x,5) = 0, & 0 < x < 10 \\ u(0,y) = 0, u(10,y) = f(y), & 0 < y < 5 \end{cases}$$

with

$$f(y) = 2\sin\frac{\pi y}{5} - 4\cos(\frac{\pi}{2} - \frac{3\pi y}{5}) + 7\sin(\pi y).$$

Find the solution $u(x,y)$. You are **NOT** required to start with separation of variables.

With $a = 10, b = 5$, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi x}{5}\sin\frac{n\pi y}{5}.$$

For $x = 10$,

$$\begin{aligned} u(10,y) =& \sum_{n=1}^{\infty} c_n \sinh(2n\pi)\sin\frac{n\pi y}{5} \\ =& 2\sin\frac{\pi y}{5} - 4\cos(\frac{\pi}{2} - \frac{3\pi y}{5}) + 7\sin(\pi y) \\ =& 2\sin\frac{\pi y}{5} - 4\sin\frac{3\pi y}{5} + 7\sin(\pi y) \end{aligned}$$

Comparing the coefficients,

$$\begin{cases} c_1\sinh(2\pi) = 2 \Rightarrow c_1 = \frac{2}{\sinh(2\pi)} \\ c_3\sinh(6\pi) = -4 \Rightarrow c_3 = -\frac{4}{\sinh(6\pi)} \\ c_5\sinh(10\pi) = 7 \Rightarrow c_5 = \frac{7}{\sinh(10\pi)} \end{cases}$$

The solution is

$$u(x,t) = \frac{2}{\sinh(2\pi)}\sinh\frac{\pi x}{5}\sin\frac{\pi y}{5} - \frac{4}{\sinh(6\pi)}\sinh\frac{3\pi x}{5}\sin\frac{3\pi y}{5} + \frac{7}{\sinh(10\pi)}\sinh(\pi x)\sin(\pi y)$$