Math 309 C Winter 2015 Final
March 16, 2015

Name:____

Student ID Number:________________________________

1	15	
2	15	
3	15	
4	15	
5	10	
Total	70	

- You have until 4:20 pm to complete the exam. There are five problems. Read all of the problems carefully before starting to work on them.
- **Turn off and put away cell phones.** You are allowed to use a scientific calculator as well as an 8.5×11 sheet of handwritten notes (you may use both sides). **No graphing calculators are allowed**
- **Show all of your work!** You will receive no credit for a problem where you just provide an answer and do not show your work. If you need more room, use the backs of pages. If you do so, please make a note for the grader (for example an arrow and a comment along the lines of "continued on back").
- You can use the back of the first and second page as the scratch paper.
- All solutions should be given in terms of **real functions (no imaginary units)**.
- If you have any questions, please raise your hand.
- Good luck!

Some formulas:

$$\int x\sin(Ax)dx = -\frac{x}{A}\cos(Ax) + \frac{1}{A}\sin(Ax) + c$$
$$\int x\cos(Ax)dx = \frac{x}{A}\sin(Ax) + \frac{1}{A}\cos(Ax) + c$$
$$\cos^2 x = \frac{1+\cos(2x)}{2}$$
$$\sin^2 x = \frac{1-\cos(2x)}{2}$$
$$\sin(2x) = 2\sin x\cos x$$

Fourier series. Each of f_a, f_b, f_c and f_d is given on the interval $(-L, L)$ and is extended to be $2L$-periodic.

$$f_a(x) = \begin{cases} -1, & \text{for } -L < x < 0 \\ 1, & \text{for } 0 < x < L \end{cases} \qquad FS = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\frac{(2k-1)\pi x}{L}.$$

$$f_b(x) = x \text{ for } -L < x < L \qquad FS = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi x}{L}.$$

$$f_c(x) = |x| \text{ for } -L < x < L \qquad FS = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(2k-1)\pi x}{L}.$$

$$f_d(x) = \begin{cases} 0, & \text{for } -L < x < -L/2 \\ 1, & \text{for } -L/2 < x < L/2 \\ 0, & \text{for } L/2 < x < L. \end{cases} \qquad FS = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\cos\frac{(2k-1)\pi x}{L}$$

Here are two functions given on $(0, L)$, and their cosine and sine series with period $2L$.

$$f_e(x) = \begin{cases} 0, & \text{for } 0 < x < L/4 \\ 1, & \text{for } L/4 < x < 3L/4 \\ 0, & \text{for } 3L/4 < x < L. \end{cases} \qquad \text{cosine series} = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}\cos\frac{(4k-2)\pi x}{L};$$

$$\text{sine series} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi/4) - \cos(3n\pi/4)}{n}\sin\frac{n\pi x}{L}$$

$$f_f(x) = \begin{cases} x, & \text{for } 0 < x < L/2 \\ L-x, & \text{for } L/2 < x < L \end{cases} \qquad \text{cosine series} = \frac{L}{4} - \frac{2L}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\frac{(4k-2)\pi x}{L}$$

$$\text{sine series} = \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}\sin\frac{(2k-1)\pi x}{L}$$

1. (15 points) Find the general solution to

$$x' = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} x + \begin{pmatrix} 3e^t \\ e^{-t} \end{pmatrix}.$$

You can use diagonalization, undetermined coefficients, or variation of parameters. CIRCLE the method you'll use.

2. Consider the homogeneous system with parameter α:

$$x' = \begin{pmatrix} -1 & 2 & 4 \\ 0 & \alpha & \alpha + 5 \\ 0 & \alpha - 5 & \alpha \end{pmatrix} x.$$

(a) (6 points) Find the general solution when $\alpha = 5$.

(b) (6 points) Find the general solution when $\alpha = 3$.

(c) (3 points) Consider $-\infty < \alpha < \infty$, for what values of α does the general solution converge to $\mathbf{0}$ as $t \to \infty$?

3. Consider an elastic string of length 20 meters with $a^2 = 9$. One end $x = 0$ is held fixed at 0, while the other end $x = 20$ is held fixed at 10. The string is set in motion with no initial velocity from the initial position $3x$ for $0 < x < 20$.

   (a) (2 points) Write the differential equation, boundary conditions, and initial conditions that the displacement $u(x, t)$ should satisfy.

   (b) (3 points) Note that the boundary conditions are nonhomogeneous, so we want to find the steady-state displacement $v(x) = \lim_{t\to\infty} u(x, t)$. Solve for $v(x)$.

   (c) (3 points) We write the solution as $u(x, t) = v(x) + w(x, t)$, find the differential equation, initial and boundary conditions that $w(x, t)$ should satisfy .

   (d) (7 points) Find the displacement $u(x, t)$. You are **NOT** required to start with separation of variables.

4. Consider the partial differential equation

$$u_{xx} = u_t + \gamma u, \qquad 0 < x < L,\ t > 0$$

with boundary conditions and initial condition

$$u_x(0,t) = 0,\ u_x(L,t) = 0, \qquad t > 0,$$
$$u(x,0) = f(x), \qquad 0 < x < L.$$

(a) (3 points) Use separation of variables, write down the differential equations that X and T must satisfy.

(b) (12 points) Find the solution $u(x, y)$.

5. (10 points) Consider the Laplace's equation

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 10,\ 0 < y < 5 \\ u(x,0) = 0, u(x,5) = 0, & 0 < x < 10 \\ u(0,y) = 0, u(10,y) = f(y), & 0 < y < 5 \end{cases}$$

with

$$f(y) = 2\sin\frac{\pi y}{5} - 4\cos(\frac{\pi}{2} - \frac{3\pi y}{5}) + 7\sin(\pi y).$$

Find the solution $u(x,y)$. You are **NOT** required to start with separation of variables.