The circumradius of a simplex via edge lengths

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Abstract

The Cayley-Menger determinant expresses the volume of a simplex in terms of its edge lengths. A similar (but less famous) identity expresses the circumradius of a simplex in terms of the top left entry of the inverse of the Cayley-Menger matrix. We present simple proofs of both identities.

An \( n\)-simplex is the convex hull of \((n + 1)\) points \(x_0, x_1, \ldots, x_n \in \mathbb{R}^n\), which are the vertices of the simplex. Points \(x_0, x_1, \ldots, x_n\) are said to be affinely independent if \(x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0\) are linearly independent.

Any simplex, \(S\), may be expressed as an affine image of a standard simplex, giving rise to the following well-known volume formula:

\[
\text{vol}(S) = \frac{1}{n!} \left| \begin{array}{c} x_1 - x_0 \\ \vdots \\ x_n - x_0 \end{array} \right|,
\]

(1)

where for a matrix \(M\) we write \(|M|\) for the absolute value of its determinant.

Note that \(\text{vol}(S) > 0\) if and only if the vertices of \(S\) are affinely independent. For such a simplex, there exists a unique point \(y \in \mathbb{R}^n\) and real number \(R > 0\) such that \(x_0, \ldots, x_n\) lie on the sphere centered at \(y\) of radius \(R\). The circumcenter of \(S\) is \(y\) and the circumradius is \(R\). We remark that \(\{y\}\) is the intersection of the hyperplanes \(H_1, H_2, \ldots, H_n\) where

\[
H_i = \{z \in \mathbb{R}^n : \|z - x_0\| = \|z - x_i\|\}, \quad 1 \leq i \leq n.
\]

(Here and throughout we write \(\|\cdot\|\) for the Euclidean distance.) This yields existence of the circumcenter. Uniqueness follows from affine independence of the vertices, since the intersection of distinct spheres is contained in a hyperplane.

For any collection of points \(z_1, \ldots, z_m \in \mathbb{R}^n\), we denote by \(D(z_1, \ldots, z_m) \in \mathbb{R}^{m \times m}\) the matrix whose entries are the squares of the pairwise distances:

\[
D(z_1, \ldots, z_m)_{ij} = \|z_i - z_j\|^2.
\]

We refer to \(D(z_1, \ldots, z_m)\) as the Euclidean distance matrix (EDM).

Recall that Heron’s formula expresses the area of a triangle in terms of its side lengths. This has a well-known generalization to \(n\) dimensions.
Proposition 1. The volume of a simplex $S$ with vertices $x_0, \ldots, x_n \in \mathbb{R}^n$ is

$$\text{vol}(S) = \frac{1}{2^{n/2}n!} \begin{vmatrix} 0 & 1 \\ 1 & D(x_0, \ldots, x_n) \end{vmatrix}^{1/2}. \quad (2)$$

The matrix appearing in Proposition 1 is called the Cayley-Menger matrix. We will present a simple proof of (2) using basic linear algebra.

In recent years, the Cayley-Menger determinant (CMD) has played a fundamental role in both pure and applied disciplines. The Bellows conjecture states that the volume of a polyhedron is invariant under flexing. This was open for over 20 years, until it was proven in 1997 using CMDs [CSW97]. Irreducibility of the CMD was established in [dS04]. The CMD is used for geometric constraint solving [MF04, SS86]. This technique has direct applications in numerous problems in robotics [RUG+09, TR05, TORC03, CAM06]. For further references see the book [DH93] on distance geometry.

Proposition 2. Let $S$ be a simplex with positive volume and vertices $x_0, \ldots, x_n$. Then its Cayley-Menger matrix is invertible and its circumradius $R$ satisfies

$$-2R^2 = \left[ \begin{array}{c} 0 \\ 1 \\ \end{array} \right] D(x_0, \ldots, x_n) \left[ \begin{array}{c} 0 \\ 1 \\ \end{array} \right] \left[ \begin{array}{c} 0 \\ 1 \\ \end{array} \right]^{-1}. \quad (3)$$

Note that the Cayley-Menger matrix is invertible by Proposition 1.

Proofs of Propositions 1 and 2. By cofactor expansion, Proposition 2 is equivalent to

$$-2R^2 = \frac{\det D(x_0, \ldots, x_n)}{\det \left[ \begin{array}{c} 0 \\ 1 \\ \end{array} \right] D(x_0, \ldots, x_n) \left[ \begin{array}{c} 0 \\ 1 \\ \end{array} \right]}. \quad (4)$$

We establish both propositions simultaneously by first computing $D(x_0, \ldots, x_n)$ and using this to compute the Cayley-Menger determinant.

As before, let $y$ denote the circumcenter of $R$. Since $D(x_0, x_1, \ldots, x_n) = D(x_0 - y, x_1 - y, \ldots, x_n - y)$, we may assume without loss of generality that $y$ is the origin, in which case $||x_i|| = R$ for all $0 \leq i \leq n$. Then $||x_i - x_j||^2 = 2R^2 - 2\langle x_i, x_j \rangle$. We write this as $-2(\langle x_i, x_j \rangle - R^2)$. Thus

$$D(x_0, \ldots, x_n) = -2(\langle x_i, x_j \rangle - R^2)_{i,j} = -2ZZ^T, \quad (5)$$

where $Z \in \mathbb{C}^{(n+1)\times(n+1)}$ is the matrix

$$Z := \begin{pmatrix} x_0 & iR \\ \vdots & \vdots \\ x_n & iR \end{pmatrix}, \quad i := \sqrt{-1}. \quad (6)$$

Subtracting the first row from all other rows in (6) yields that

$$\det Z = \det \begin{pmatrix} x_0 & iR \\ x_1 - x_0 & 0 \\ \vdots & \vdots \\ x_n - x_0 & 0 \end{pmatrix} = \pm iR n! \text{vol}(S). \quad (7)$$
Thus substituting (7) into the determinant of (5) yields that
\[ \det D(x_0, \ldots, x_n) = -R^2 \left( n! \text{vol}(S) \right)^2 \cdot (-2)^{n+1}. \] (8)

By cofactor expansion along the first column,
\[
\begin{vmatrix}
0 & 1 \\
1 & D(x_0, \ldots, x_n)
\end{vmatrix} = \det \left( \frac{1}{2R^2} \begin{bmatrix}
1 & D(x_0, \ldots, x_n)
\end{bmatrix} \right) - \det \left( \frac{1}{2R^2} \begin{bmatrix}
0 & D(x_0, \ldots, x_n)
\end{bmatrix} \right) \] (9)

The former determinant in (9) vanishes, since the factorization in (5) yields that
\[
\left( \frac{1}{2R^2} \begin{bmatrix}
1 & D(x_0, \ldots, x_n)
\end{bmatrix} \right) = -2 \begin{pmatrix}
0 & 0 & -\frac{1}{2R} \\
0 & x_0 & iR \\
\vdots & \vdots & \vdots \\
0 & x_n & iR
\end{pmatrix} \begin{pmatrix}
0 & 0 & -\frac{1}{2R} \\
0 & x_0 & iR \\
\vdots & \vdots & \vdots \\
0 & x_n & iR
\end{pmatrix}^T.
\]

Since the determinant of this matrix vanishes, (9) yields that
\[
\det \left( \begin{bmatrix}
0 & 1 \\
1 & D(x_0, \ldots, x_n)
\end{bmatrix} \right) = -\frac{1}{2R^2} D(x_0, \ldots, x_n),
\]
implying (4) and therefore Proposition 2. Finally, combining (4) with (8) yields Proposition 1. \qed

References


[dS04] Carlos d’Andrea and Martin Sombra. The cayley-menger determinant is irreducible for \( n \geq 3 \). 2004.


