The circumradius of a simplex via edge lengths

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Abstract

The Cayley-Menger determinant expresses the volume of a simplex in terms of its edge lengths. A similar (but less famous) identity expresses the circumradius of a simplex in terms of the top left entry of the inverse of the Cayley-Menger matrix. We present simple proofs of both identities.

An ***n*-simplex** is the convex hull of $(n+1)$ points $x_0, x_1, \ldots, x_n \in \mathbb{R}^n$, which are the **vertices** of the simplex. Points $x_0, x_1, \ldots, x_n$ are said to be **affinely independent** if $x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0$ are linearly independent.

Any simplex, S, may be expressed as an affine image of a standard simplex, giving rise to the following well-known volume formula:

$$\mathrm{vol}(S) = \frac{1}{n!} \begin{vmatrix} x_1 - x_0 \\ \vdots \\ x_n - x_0 \end{vmatrix}, \tag{1}$$

where for a matrix M we write $|M|$ for the absolute value of its determinant.

Note that $\mathrm{vol}(S) > 0$ if and only if the vertices of S are affinely independent. For such a simplex, there exists a unique point $y \in \mathbb{R}^n$ and real number $R > 0$ such that $x_0, \ldots, x_n$ lie on the sphere centered at y of radius R. The **circumcenter** of S is y and the **circumradius** is R. We remark that $\{y\}$ is the intersection of the hyperplanes $H_1, H_2, \ldots, H_n$ where

$$H_i = \{z \in \mathbb{R}^n \colon \|z - x_0\| = \|z - x_i\|\}, \qquad 1 \le i \le n.$$

(Here and throughout we write $\|\cdot\|$ for the Euclidean distance.) This yields existence of the circumcenter. Uniqueness follows from affine independence of the vertices, since the intersection of distinct spheres is contained in a hyperplane.

For any collection of points $z_1, \ldots, z_m \in \mathbb{R}^n$, we denote by $D(z_1, \ldots, z_m) \in \mathbb{R}^{m \times m}$ the matrix whose entries are the squares of the pairwise distances:

$$D(z_1, \ldots, z_m)_{ij} = \|z_i - z_j\|^2.$$

We refer to $D(z_1, \ldots, z_m)$ as the Euclidean distance matrix (EDM).

Recall that Heron's formula expresses the area of a triangle in terms of its side lengths. This has a well-known generalization to n dimensions.

Proposition 1. *The volume of a simplex S with vertices $x_0, \ldots, x_n \in \mathbb{R}^n$ is*

$$\operatorname{vol}(S) = \frac{1}{2^{n/2} n!} \begin{vmatrix} 0 & 1 \\ 1 & D(x_0, \ldots, x_n) \end{vmatrix}^{1/2}. \tag{2}$$

The matrix appearing in Proposition 1 is called the Cayley-Menger matrix. We will present a simple proof of (2) using basic linear algebra.

In recent years, the Cayley-Menger determinant (CMD) has played a fundamental role in both pure and applied disciplines. The Bellows conjecture states that the volume of a polyhedron is invariant under flexing. This was open for over 20 years, until it was proven in 1997 using CMDs [CSW97]. Irreducibility of the CMD was established in [dS04]. The CMD is used for geometric constraint solving [MF04, SS86]. This technique has direct applications in numerous problems in robotics [RUG$^+$09, TR05, TORC03, CAM06]. For further references see the book [DH93] on distance geometry.

Proposition 2. *Let S be a simplex with positive volume and vertices $x_0, \ldots, x_n$. Then its Cayley-Menger matrix is invertible and its circumradius R satisfies*

$$-2R^2 = \left[\begin{pmatrix} 0 & 1 \\ 1 & D(x_0, \ldots, x_n) \end{pmatrix}^{-1} \right]_{1,1}. \tag{3}$$

Note that the Cayley-Menger matrix is invertible by Proposition 1.

Proofs of Propositions 1 and 2. By cofactor expansion, Proposition 2 is equivalent to

$$-2R^2 = \frac{\det D(x_0, \ldots, x_n)}{\det \begin{pmatrix} 0 & 1 \\ 1 & D(x_0, \ldots, x_n) \end{pmatrix}}. \tag{4}$$

We establish both propositions simultaneously by first computing $D(x_0, \ldots, x_n)$ and using this to compute the Cayley-Menger determinant.

As before, let y denote the circumcenter of R. Since $D(x_0, x_1, \ldots, x_n) = D(x_0 - y, x_1 - y, \ldots, x_n - y)$, we may assume without loss of generality that y is the origin, in which case $\|x_i\| = R$ for all $0 \leq i \leq n$. Then $\|x_i - x_j\|^2 = 2R^2 - 2\langle x_i, x_j \rangle$. We write this as $-2(\langle x_i, x_j \rangle - R^2)$. Thus

$$D(x_0, \ldots, x_n) = -2(\langle x_i, x_j \rangle - R^2)_{i,j} = -2ZZ^T, \tag{5}$$

where $Z \in \mathbb{C}^{(n+1)\times(n+1)}$ is the matrix

$$Z := \begin{pmatrix} x_0 & \boldsymbol{i}R \\ \vdots & \vdots \\ x_n & \boldsymbol{i}R \end{pmatrix}, \qquad \boldsymbol{i} := \sqrt{-1}. \tag{6}$$

Subtracting the first row from all other rows in (6) yields that

$$\det Z = \det \begin{pmatrix} x_0 & \boldsymbol{i}R \\ x_1 - x_0 & 0 \\ \vdots & \vdots \\ x_n - x_0 & 0 \end{pmatrix} = \pm\, \boldsymbol{i}R\, n! \operatorname{vol}(S). \tag{7}$$

Thus substituting (7) into the determinant of (5) yields that

$$\det D(x_0, \ldots, x_n) = -R^2 \left(n!\,\mathrm{vol}(S)\right)^2 \cdot (-2)^{n+1}. \tag{8}$$

By cofactor expansion along the first column,

$$\begin{aligned} &\det \begin{pmatrix} 0 & 1 \\ 1 & D(x_0, \ldots, x_n) \end{pmatrix} \\ &\qquad = \det \begin{pmatrix} \frac{1}{2R^2} & 1 \\ 1 & D(x_0, \ldots, x_n) \end{pmatrix} - \det \begin{pmatrix} \frac{1}{2R^2} & 1 \\ 0 & D(x_0, \ldots, x_n) \end{pmatrix} \end{aligned} \tag{9}$$

The former determinant in (9) vanishes, since the factorization in (5) yields that

$$\begin{pmatrix} \frac{1}{2R^2} & 1 \\ 1 & D(x_0, \ldots, x_n) \end{pmatrix} = -2 \begin{pmatrix} 0 & 0 & -\frac{1}{2\boldsymbol{i}R} \\ 0 & x_0 & \boldsymbol{i}R \\ \vdots & \vdots & \vdots \\ 0 & x_n & \boldsymbol{i}R \end{pmatrix} \begin{pmatrix} 0 & 0 & -\frac{1}{2\boldsymbol{i}R} \\ 0 & x_0 & \boldsymbol{i}R \\ \vdots & \vdots & \vdots \\ 0 & x_n & \boldsymbol{i}R \end{pmatrix}^T .$$

Since the determinant of this matrix vanishes, (9) yields that

$$\det \begin{pmatrix} 0 & 1 \\ 1 & D(x_0, \ldots, x_n) \end{pmatrix} = -\frac{1}{2R^2} D(x_0, \ldots, x_n),$$

implying (4) and therefore Proposition 2. Finally, combining (4) with (8) yields Proposition 1.

□

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