Biregular networks

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Abstract

We construct biregular networks that are at least n to 1.

Contents

1 Bipartite Networks

Consider an electrical network Γ on a graph G. The nodes of G are always partitioned into a boundary set (∂V) and interior set ($\operatorname{Int} V$).

Definition 1. G is **bipartite** if there are no $\partial - \partial$ or $\operatorname{Int} - \operatorname{Int}$ edges.

Bipartite networks have less complex response matrices than general networks. Start by considering the Kirchhoff matrix

$$K = \begin{bmatrix} A & -B \\ -B^T & D \end{bmatrix}.$$

The bipartite condition is equivalent to forcing A, D to be diagonal. Each of A, D are determined from B by the 0-sum condition, so B encodes all of the network information. For bipartite graphs, the B matrix is more fundamental than the Kirchhoff matrix.

Definition 2. To each bipartite network Γ, associate the $|\partial V| \times |\operatorname{Int} V|$ matrix $B(\Gamma)_{ij} = \gamma_{ij}$.

Note that $B(\Gamma)$ consists entirely of nonnegative entries, as opposed to the Kirchhoff matrix. Moreover, $B(\Gamma)_{ij}$ is positive if and only if ∂-node i and Int-node j are incident.

Definition 3. Let $\mathcal{O}(n, m)$ denote the set $n \times m$ matrices with nonnegative entries. Say that two matrices in $\mathcal{O}(n, m)$ are **graph equivalent** if they have the same zero entries. An **arrangement** is an equivalence class of $\mathcal{O}(n, m)$.

Definition 4. If Γ is a bipartite network with underlying graph G, let $\mathcal{O}(G)$ denote the equivalence class of $B(\Gamma)$.

Claim 1. *The assignment $\Gamma \to B(\Gamma)$ is a bijection from the family of bipartite networks to $\mathcal{O}(n, m)$. The assignment $G \to \mathcal{O}(G)$ is a bijection from the family of bipartite graphs to the arrangements of $\mathcal{O}(n, m)$.*

1.1 The inverse problem for bipartite networks

Definition 5. Let σ_j denote the sum of column j in $B(\Gamma)$; that is,

$$\sigma_j = \sum_{i=1}^{|\partial V|} \gamma_{ij}, \qquad 1 \leq j \leq |\operatorname{Int} V|.$$

These entries give rise to the $|\operatorname{Int} V| \times |\operatorname{Int} V|$ diagonal matrix $D(\Gamma)$, given by $D_{jj} = \sigma_j$.

Definition 6. Consider the map

$$\Phi\colon \mathcal{O}(|\partial V| \times |\operatorname{Int} V|) \to \mathcal{O}(|\partial V| \times |\partial V|), \qquad B \mapsto BD^{-1}B^T.$$

The **bipartite response map** consists of the upper triangular entries of Φ:

$$B \mapsto \{\Phi(B)_{ij}\}_{i<j}$$

This definition corresponds to the usual definition for electrical networks, since $\Lambda = A - BD^{-1}B^T$ is determined by its upper triangular entries.

Definition 7. Γ is n **to 1** if the response map has largest fiber of size n. If the map has some fiber of size n, we say that Γ is **at least** n **to 1**. If Γ is 1 to 1, we say that Γ is **recoverable**.

2 Biregular Networks

Definition 8. A bipartite graph G is **biregular** if each interior node has b neighbors and each boundary node has d neighbors, for some b, d.

(a_b, c_d) denotes the family of all biregular graphs with $|\operatorname{Int} V| = a$,$|\partial V| = c$, and b, d as above.

Note that if $G \in (a_b, c_d)$, then G has a total of $ab = cd$ edges. Sometimes, we shorten $G \in (a_b, c_d)$ by simply writing $G(a_b, c_d)$.

Definition 9. If Γ is a network with underlying biregular graph $G(a_b, c_d)$, we say that $\Gamma(a_b, c_d)$ is a **biregular network**.

Example 10. Some biregular networks:

1. An n-star is $(1_n, n_1)$
2. The triangle-in-triangle is $(3_4, 6_2)$ and is 2 to 1 (French, 2004)
3. The complete quadrangle is $(4_3, 6_2)$ and is 1 to 1 (Reichert, 2004)
4. The "flux capacitor" graphs belong to $(3_{2m}, 3m_2)$; $m = 2$ is the triangle-in-triangle.
5. Similar to the flux capacitors are the star-in-star graphs

$$S_n^l \in \left(n_{\binom{n-1}{l-1}}, \binom{n}{l}_l \right).$$

To construct S_n^l, start with n interior nodes. For each choice of l interiors, place a boundary node incident to all of them.

2.1 Multiplication of Biregular Networks

Definition 11. Consider a network $\Gamma(a_b, c_d)$. Replace each boundary node with n copies of itself, each having the same incidences and conductivities as the original. The resulting network is called $n\Gamma$.

Claim 2 (Properties of multiplication)**.** *$n\Gamma \in (a_{nb}, nc_d)$ and*

$$B(n\Gamma) = B(\Gamma) \otimes \left.\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right\} n$$

The symbol $\otimes$ denotes the Kronecker product of matrices (it corresponds to tensor product). Multiplication replaces the entries of B with $n \times 1$ block matrices.

Example 12. Lots of graphs are multiples:

1. The flux capacitors can be written as mS_3^2.
2. Generalizing the previous example are the graphs mS_n^l, of the form

$$\left(n_{m\binom{n-1}{l-1}}, m\binom{n}{l}_l\right)$$

3. The n-gon in n-gon is $2G$, where $G(n_2, n_2)$ is the cyclic series graph.
4. Numerous examples from (French, 2004) and (Reichert, 2004).

It turns out that multiplication can improve recoverability of a network. For instance S_3^2 is a cyclic series, so is ∞ to 1. However, the triangle-in-triangle graph is $2S_3^2$, which is 2 to 1.

Claim 3 (∞-bound)**.** *Suppose $c < 2d + 1$. Then every $G(a_b, c_d)$ is ∞ to 1.*

Proof. There are cd unknowns, but only $\binom{c}{2}$ equations (see Reichert, 2004). $\square$

This criterion shows that many biregular graphs are superficially uninteresting, because they exhibit ∞ to 1 behavior. However, multiplication increases c while leaving d fixed; thus we can use multiplication as a tool to get around ∞ to 1 behavior.

Claim 4. *Facts about multiplication.*

1. $D(n\Gamma) = nD(\Gamma)$
2. *In the following line, the rightmost matrix is $n \times n$.*

$$BD^{-1}B^T(n\Gamma) = BD^{-1}B^T(\Gamma) \otimes \begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

Proof. 1. Straightforward

2. Set $B = B(\Gamma)$ and $D = D(\Gamma)$. Using properties of the Kronecker product,

$$\begin{aligned} BD^{-1}B^T(n\Gamma) &= \left(B \otimes \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}\right)(nD)^{-1}\left(B \otimes \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}\right)^T \\ &= \left(B \otimes \begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}\right)\left(D^{-1} \otimes \left(\frac{1}{n}\right)\right)\left(B^T \otimes \begin{pmatrix}1 & \cdots & 1\end{pmatrix}\right) \\ &= BD^{-1}B^T \otimes \begin{pmatrix}\frac{1}{n} & \cdots & \frac{1}{n}\\ \vdots & \ddots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n}\end{pmatrix} \end{aligned}$$

$\square$

3 Construction of R_n

Suppose we have a biregular network with n^2 boundaries, each incident to $n(n-1)$ interiors. Observe that

$$n^2 < 2n(n-1) + 1 \leftrightarrow (n-1)^2 > 0.$$

Thus by the previous claim, such a network is ∞ to 1.

Claim 5. *For $n > 1$, there is a biregular graph $R_n(n^2_{n(n-1)}, n^2_{n(n-1)})$, whose symmetries alone force it to be n to 1 or greater.*

By the previous remark, such a graph is always ∞ to 1, so the bound alone is uninteresting. However, the interest comes by examining the symmetries of R_n. Later we will see that the same argument extends to mR_n, which need not be ∞ to 1.

We construct matrices $B^{(1)}, \cdots, B^{(n)}$ that are "B" matrices for a biregular graph R_n. The matrices will all yield the same "D" matrix, which turns out to be a scalar multple of the identity. Then we prove that the matrix

$$B^{(k)}(B^{(k)})^T$$

is independent of k, from which it follows that $B^{(1)}, \cdots, B^{(n)}$ have the same response matrix.

Setup. For any integer a, let $[a]_m$ denote the unique integer such that

$$a \equiv [a]_m \mod m, \ 1 \leq [a]_m \leq m.$$

Let $B^{(k)}$ for $1 \leq k \leq n$ denote the $n^2 \times n^2$ matrix with entries

$$B^{(k)}_{ij} = \begin{cases} 0, & i \equiv j \mod n \\ \left[\left\lfloor \frac{i-j}{n} \right\rfloor + k\right]_n, & \textit{else.} \end{cases}$$

For example when $n = 3$,

$$B^{(1)} = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 & 2 & 0 & 3 & 3 \\ 3 & 0 & 1 & 1 & 0 & 2 & 2 & 0 & 3 \\ 3 & 3 & 0 & 1 & 1 & 0 & 2 & 2 & 0 \\ 0 & 3 & 3 & 0 & 1 & 1 & 0 & 2 & 2 \\ 2 & 0 & 3 & 3 & 0 & 1 & 1 & 0 & 2 \\ 2 & 2 & 0 & 3 & 3 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 & 3 & 3 & 0 & 1 & 1 \\ 1 & 0 & 2 & 2 & 0 & 3 & 3 & 0 & 1 \\ 1 & 1 & 0 & 2 & 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

Since $B^{(1)}, \cdots, B^{(n)}$ have the same configuration of 0 entries, they all correspond to the same network R_n (technically it is $-B^{(k)}$ that corresponds to the B matrix of the network, but we ignore this detail). This network has the form $(n^2_{n(n-1)}, n^2_{n(n-1)})$ since there are n^2 rows and columns, each with $n(n-1)$ non-zero entries.

A short calculation shows that each column of $B^{(k)}$ sums to

$$\frac{(n-1)n(n+1)}{2},$$

which means that D is a scalar multiple of the identity. Hence we may work with the map $B \mapsto BB^T$ instead of $BD^{-1}B^T$. $\square$

Subclaim 5.1. *$B^{(k)}(B^{(k)})^T$ is independent of k.*

Proof. We show that the i,j entry of $B^{(k)}(B^{(k)})^T$ is independent of k. Indeed,

$$\begin{aligned}
[B^{(k)}(B^{(k)})^T]_{ij} &= \sum_{l=1}^{n^2} B^{(k)}_{il} B^{(k)}_{jl} \\
&= \sum_{\substack{l=1 \\ l \not\equiv i,j \mod n}}^{n^2} \left[\left\lfloor \frac{i-l}{n} \right\rfloor + k\right]_n \left[\left\lfloor \frac{j-l}{n} \right\rfloor + k\right]_n \\
(\tilde{l} = l - nk, l \equiv \tilde{l}) \qquad &= \sum_{\substack{\tilde{l}=1-nk \\ \tilde{l} \not\equiv i,j \mod n}}^{n^2-nk} \left[\left\lfloor \frac{i-\tilde{l}}{n} \right\rfloor\right]_n \left[\left\lfloor \frac{j-\tilde{l}}{n} \right\rfloor\right]_n \\
&= \sum_{\substack{\tilde{l}=1 \\ \tilde{l} \not\equiv i,j \mod n}}^{n^2} \left[\left\lfloor \frac{i-\tilde{l}}{n} \right\rfloor\right]_n \left[\left\lfloor \frac{j-\tilde{l}}{n} \right\rfloor\right]_n .
\end{aligned}$$

In the last line, we have manipulated the index of the summation by sending $\tilde{l} \mapsto \tilde{l} + n^2$ when $\tilde{l} < 1$. Note that this transformation leaves the summand unchanged. The final expression is independent of k, as desired. $\square$

Corollary 13. *For each $m, n > 1$, the graph $mR_n(n^2_{mn(n-1)}, mn^2_{n(n-1)})$ is at least n to 1.*

Proof. Consider any network Γ on R_n. By Claim 4,

$$BD^{-1}B^T(m\Gamma) = BD^{-1}B^T(\Gamma) \otimes M,$$

where M is the $m \times m$ matrix filled with all entries $\frac{1}{m}$. Thus all the matrices $B^{(k)}$ we constructed above still lead to the same response matrix. By the same argument, mR_n is at least n to 1. $\square$

Note that mR_n no longer satisfies the ∞-bound, since

$$mn^2 \geq 2n^2 \geq 2n(n-1) + 1.$$

4 Questions about the mR_n family

This construction shows that biregular networks are a good place to look for n to 1 behavior. Questions that would be fun to work on with REU students:

1. What do the graphs mR_n look like? What are their star-K transforms?
2. Are these networks finite to 1? How can we establish upper bounds?
3. How do these graphs relate to the work of French and Reichert?
4. What is the genus of mR_n?
5. Can Reid's algorithm give insight into the fiber sizes of mR_n for small m, n?
6. Are there ∞ to 1 graphs that people have overlooked which become n to 1 when multiplied?

4.1 Partial Answers

1. We've developed methods in Mathematica to render biregular graphs. Their star-K transforms are better understood now.
2. Katherine took star-K transforms of various biregular graphs. She realized that an n-to-1 graph needs to have both single edges and n-fold edges in its star-K transform. Many of the graphs we had been looking at have way too many edges to satisfy this criterion, including most of the mR_n graphs. So the answer to this question is "No".
3. See the projective geometry section, which will be fleshed out.
4. These graphs are less interesting now, so we should move our genus efforts to different graphs.
5. Reid's algorithm appears to run too slowly to be of practical use.
6. This question is still open and potentially interesting.

Continuing from the negative answer to question 2, we started by looking at biregular graphs that have single and triple edges in their star-K. The simplest case is the $3\partial(K_4)$ graph, which we analyzed. The family of graphs $n\partial(K_{n+1})$ all have single and n-fold edges, so they are candidates for n-to-1 behavior. Unfortunately, I suspect they are 1-to-1. There is a lot that I should write up here.

5 Projective Geometry

My interest in biregular networks is due to their connection with projective geometry. Coincidentally, a related family of graphs (called flowers) were studied in 2004 by Reichert and French.

Definition 14. An **incidence relation** is a triple (P, L, I), where P is a set of points, L is a set of lines, and $I \subset P \times L$ is a relation between points and lines.

Given any point, the incidence relation tells us which lines it belongs to. Given a line, the incidence relation determine the points on that line. The last two sentences are an example of **duality** (interchanging the roles of lines and points). Theorems of projective geometry remain true under such transformations.

Definition 15. An **abstract configuration** of the form (a_b, c_d) consists of a lines and c points, each incident to b points and d lines respectively.

Replacing points and lines yields a dual configuration, of the form (c_d, a_b).

Definition 16. A **projective configuration** is an abstract configuration such that every pair of points is incident to a single line.

5.1 Connection to biregular graphs

Claim 6. *There is a bijection between abstract configurations and biregular graphs, called the* **Levi graph construction**.

Definition 17. The **girth** of a graph is the length of its smallest cycle.

Note that the girth of a bipartite graph is even and > 2.

Claim 7. *The following are equivalent for a biregular graph G:*

(a) G corresponds to a projective configuration

(b) The girth of G is > 4

(c) The $\star - K$ transform of G has no double edges.

If any of the above conditions hold, G is said to be **projective**. As a consequence of (c), projective networks are recoverable (provided $b > 2$).

There is an important qualification that I'm not quite sure how to phrase correctly yet. We can extend the $\star - K$ transform such that it is meaningful from a graph-theoretic standpoint, even when the interior nodes have degree < 3. However, the transform is no longer electrically equivalent for degree < 3. Hence the qualification.

A special case of this statement was discovered in (Reicher, 2004): the complete quadrangle $(4_3, 6_2)$ was shown to be recoverable. In fact, Reichert showed that the complete quadrangle is the smallest recoverable flower.

6 Cactus Graphs

These are symmetric bipartite graphs that appear to exhibit n-to-1 behavior. Unfortunately, they are not biregular. More to come.