On the arity of a graph

Avi Levy

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1 Introduction

Consider a graph with boundary $G = (\mathrm{Int}\, G, \partial G)$. Placing positive edge weights on G produces an electrical network. In this context, we introduce the arity $\kappa(G)$. It is an "electric" graph invariant which measures the complexity of the possible networks arising from G.

When $\kappa(G) = n$ is finite, the graph G is said to be "strongly n-to-1". Broadly speaking, the "n-to-1 problem" asks if there are graphs which are strongly n-to-1 for any given n. The problem was answered in the affirmative by several recent constructions.

In this paper, we present a general framework for determining κ. This method is powerful enough to produce a plethora of novel solutions to the n-to-1 problem. The primary new ingredient is a diagram algebra that encodes electrical properties of graphs.

1.1 Precise definitions

In this section, G is always a finite graph with (possibly empty) boundary.

Definition 1.1. *The* **response map** *of* G*,*

$$L_G : \mathbb{R}_{>0}^{|E|} \to M(|\partial G|, |\partial G|),$$

sends a conductivity vector to its response matrix (see appendix).

Definition 1.2. *The* **arity** $\kappa(G)$ *is*

$$\kappa(G) = \sup\{|L_G^{-1}(a)| : a \in M(|\partial G|, |\partial G|)\},$$

i.e. $\kappa(G)$ *is the cardinality of the largest fiber of* L_G*.*

A breakthrough result in [CIM98] showed that when G is circular planar, $\kappa \in \{1, \mathfrak{c}\}$. In this case, $\kappa = 1$ precisely when G is critical. It was unknown if κ could take any other finite values (for general G) until [Esser] discovered that $2 \in \mathrm{Im}(\kappa)$. In [French] it was shown that $2^n \in \mathrm{Im}(\kappa)$, and in [Klumb] it was shown that $\mathbb{N} \subset \mathrm{Im}(\kappa)$. Most recently, [Dale] discovered that

$$\mathrm{Im}(\kappa) = \mathbb{N} \cup \{\mathfrak{c}\},$$

where $\mathfrak{c}$ is the cardinality of the continuum, completing the determination of $\mathrm{Im}(\kappa)$.

1.2 Overview

We introduce a general family of graphs (with boundary) and compute their arity. The graphs are built out of units called "quadrilaterals" and "switches". To each such graph G, there is an associated "loop diagram" which describes the mechanics of the graph. The loop diagram is a directed graph whose vertices are assigned indeterminates and whose arrows are assigned "arms". Analyzing the arity of the original graph corresponds to analyzing the interaction of arms in the loop diagram. This is further reduced to the study of the "traces" of the arms.

We compute the arity $\kappa(G)$ by providing lower and upper bounds which agree. The lower bound consists of an algorithm which constructs many sets of conductivities for the network which yield the same response map. The upper bound estimates the number of fixed points of a circle map using topological techniques.

Start with G: suppose its loop diagram contains a single vertex (so we are working with a generalized cactus graph).

Upper bound:

(a) Produce the associated singularity plot on $\mathbb{S}^1$ and designate an interval U

(b) Count number of sign alternations.

Lower bound:

(a) Using the sign alternations, compute the relative spacing of the roots and singularities.

(b) Fix an interval U, fix the set of "valid" roots inside of it, fix the remaining roots and singularities.

(c) Compute the interpolation as a sum of factorizable arms.

(d) Factorize the factorizable arms to obtain intermediate conductivities.

(e) Populate the graph with this data.

This draft currently discusses only the factorization part of the lower bound. Fortunately, this is the heart of the entire arity computation.

2 Arm theory

Arms are the main object of study in this paper. They are a formal abstraction of the manner in which information propagates through an electrical network, and they keep track of positivity conditions. Before introducing them, we begin with some basic topological preliminaries.

2.1 Topology of intervals on $\mathbb{S}^1$

Let $\overline{\mathbb{R}}$ denote the one-point compactification of $\mathbb{R}$, which we identify with $\mathbb{S}^1$. When drawing diagrams, values increase as we travel counterclockwise.

Definition 2.1. *Given* $a, b \in \overline{\mathbb{R}}$*, we define the* **circular interval** (a, b) *as the open arc which begins at* a *and increases to* b*.*

When $a, b \in \mathbb{R}$ and $a < b$, this corresponds to usual open interval $(a, b) \subset \mathbb{R}$. However if $b < a$, the interval becomes

$$(b, a) = (b, \infty) \cup \{\infty / -\infty\} \cup (-\infty, a).$$

In general, observe that when $a \neq b$ the sets (a, b) and (b, a) are non-empty and disjoint. We operate exclusively in the topological space $\mathbb{S}^1$ (and take closures with respect to this space). In particular, $\overline{(0, \infty)} = [0, \infty]$.

Let f denote a homeomorphism of $\mathbb{S}^1$. Recall the **topological degree** (or winding number) of f, denoted by $\deg f$. It is a homomorphism from the group of homeomorphisms to $\mathbb{Z}$, and as such is ± 1. For us, $\deg f$ reports whether f preserves or reverses orientation of circular intervals.

Lemma 2.2. *Let* (a, b) *be a circular interval. Then*

$$f((a,b)) = \begin{cases} (f(a), f(b)) & \deg f = 1 \\ (f(b), f(a)) & \deg f = -1 \end{cases}$$

We also record intersection properties of intervals. Let $[[a_1, \cdots, a_n]]$ denote the condition that the elements $a_1, \cdots, a_n$ appear in counterclockwise circular order.

Lemma 2.3. *Let* (a, b) *and* (c, d) *be circular intervals. Then*

$$(a, b) \cap (c, d) = \begin{Bmatrix} (a, d) & [[a, d, b]] \\ \varnothing & \textit{else} \end{Bmatrix} \cup \begin{Bmatrix} (b, c) & [[c, b, d]] \\ \varnothing & \textit{else} \end{Bmatrix}$$

Corollary 2.4. $(a, b) \cap (b, c) = (a, c)$ *when* $[[c, b, a]]$ *and is empty otherwise.*

2.2 What is an arm?

Definition 2.5. *An* **arm** *is a pair* (f, U) *where* f *is a homeomorphism of* $\overline{\mathbb{R}}$ *and* $U \subset \mathbb{R}^+ \cap f^{-1}\mathbb{R}^+$.

Every homeomorphism f can be "lifted" to an arm by choosing $U = \mathbb{R}^+ \cap f^{-1}\mathbb{R}^+$. In this case the arm is called **maximal** and is denoted by (f). There should be no confusion with stray parentheses, since it is clear from context whether we are working with a function: f, or with an arm: (f). In particular, there is an arm corresponding to every homeomorphism of $\overline{\mathbb{R}}$.

Arms are a chimera of functions and sets, and therefore inherit operations from each.

Definition 2.6 (Arms as sets). *If* $\mu = (f, U)$, *the set* U *is called the* **domain [of positivity]** *of* μ *and is denoted by* $U = \mathrm{Dom}(\mu)$. *If* μ, ν *are arms with the same function, we write* $\mu \subset \nu$ *whenever* $\mathrm{Dom}(\mu) \subset \mathrm{Dom}(\nu)$.

In this case, μ is called a **restriction** of ν. Note that $\mathrm{Dom}(\mu)$ is not the domain of f. In fact, they can never agree (since the domain of f contains ∞). Set adjectives applied to arms refer to their domains. For instance, an arm μ is **connected** whenever $\mathrm{Dom}(\mu)$ is connected.

When working with arms, it is useful to be able to extend the domain of positivity.

Definition 2.7. *An arm* $\mu = (f, U)$ *is* **regular** *whenever* $\overline{U} \subset \mathbb{R}^+ \cap f^{-1}(\mathbb{R}^+)$.

In other words, $(f, \overline{U}) \subset (f)$. Since (f) has open domain, there is extra "room" around the closed set $\overline{U}$ that we can work with (this will end up being used for extra singularities). The terminology comes from manifold theory.

Definition 2.8 (Arms as functions). *If* $\mu = (f, U)$ *and* V *is any set, we define* $\mu V = f^{-1}V$.

The reason we use f^{-1} is to form an action (see Lemma 2.10 (d)). We say an arm is "Möbius" if its function is Möbius (and similarly for other adjectives). The parity of an arm is denoted by $\deg \mu$ and is the topological degree $\deg f$ of the circle map f.

The most important property of arms is that they can be concatenated.

Definition 2.9 (Arm concatenation).

$$(f, U) \rtimes (g, V) = (g \circ f, U \cap f^{-1}(V)).$$

By way of motivation, consider the sequence

$$x \mapsto f(x) \mapsto g(f(x)).$$

The conditions $x > 0$ and $f(x) > 0$ are equivalent to specifying the maximal arm (f). The conditions $x > 0, g(f(x)) > 0$ are equivalent to specifying the maximal arm $(g \circ f)$. Including all three conditions $x > 0, f(x) > 0, g(f(x)) > 0$ yields the arm concatenation $(f) \rtimes (g)$.

Lemma 2.10. *Suppose μ, ν, λ are arms and V is a set. Then:*

(a) $\mu \rtimes \nu$ *is an arm*

(b) $(\mathrm{id}) \rtimes \mu = \mu \rtimes (\mathrm{id}) = \mu$

(c) $\mu \rtimes (\nu \rtimes \lambda) = (\mu \rtimes \nu) \rtimes \lambda$

(d) $\mu \rtimes (\nu V) = (\mu \rtimes \nu)V$ *and* $(\mathrm{id})V = V$

(e) $\mu \subset \nu \implies \lambda \rtimes \mu \subset \lambda \rtimes \nu, \mu \rtimes \lambda \subset \nu \rtimes \lambda$

Remark 2.11. *Meaning of Lemma 2.10:*

(a)-(c) These properties state that $(arms, \rtimes)$ *is a monoid.*

(d) states that arms act on sets.

(e) states that the poset $(arms, \subset)$ *is translation invariant.*

The identity arm is $(\mathrm{id}) = (\mathrm{id}, \mathbb{R}^+)$*, the maximal arm corresponding to the identity function.*

The concatenation of maximal arms need not be maximal.

2.3 Arm classes

Consider the free monoid $\mathcal{M} = \langle a, b \rangle$. Elements of this monoid are formal words in a, b. If $w \in \mathcal{M}$, we let $|w|$ denote its length. To each formal word we associate a class of arms as follows:

(a) $\mathrm{id} \in \mathcal{M}$ is identified with the singleton class $\{(\mathrm{id})\}$

(b) $a \in \mathcal{M}$ is identified with $\{(* - x) : * \in \mathbb{R}^+\}$

(c) $b \in \mathcal{M}$ is identified with $\{(*/x) : * \in \mathbb{R}^+\}$

(d) Given words $w_1, w_2 \in \mathcal{M}$, the word $w = w_1 w_2$ is identified with the class

$$\{\mu_1 \rtimes \mu_2 : \mu_1 \in w_1, \mu_2 \in w_2\}$$

Our identification is well-defined due to associativity of $\rtimes$. Note that all elements of id, a, b are clearly maximal. This does not hold for the higher arm classes. In fact, "most" elements of the larger arm classes have empty domain (roughly speaking, the coefficients must be chosen rather delicately to ensure a domain of positivity). In general, a word w is associated with a $|w|$-parameter family of arms.

For instance when $w = aba^2b$, the associated class is the 5-parameter family

$$\{(d_1 - x) \rtimes (d_2/x) \rtimes (d_3 - x) \rtimes (d_4 - x) \rtimes (d_5/x) : d_i \in \mathbb{R}^+\}.$$

As always, the parentheses indicate the presence of maximal arms.

Generally speaking, arm concatenation shrinks the domain of an arm and is therefore irreversible. This explains why we must work in the monoid $\mathcal{M}$, instead of in a group. However, $\mathcal{M}$ admits a useful reduction that restores reversibility.

Definition 2.12. *The* **reduction map** *is the surjective monoid homomorphism* π *defined by*

$$\pi\colon \mathcal{M} \to D_\infty, \qquad a^2, b^2 \mapsto \mathrm{id},$$

where D_∞ *is the infinite dihedral group.*

For a word $w \in \mathcal{M}$ we let $\overline{w} = \pi(w)$ denote its reduction. We naturally identify D_∞ with a subset of $\mathcal{M}$, referred to as the **alternating (or reduced) words** of $\mathcal{M}$. Elements of the associated arm classes are referred to as **alternating arms**. Lastly, we say that an alternating word is **non-degenerate** if it has at least 2 instances of a. Later we will see that non-degenerate alternating words are "universal".

Example 2.13. *The word* $(ab)^n$ *is alternating, and therefore reduced. On the other hand,*

$$\overline{(ab^2)(ab)^n(ab^2)} = (ba)^n.$$

The degenerate alternating words are $\mathrm{id}, a, b, ab, ba, bab$.

$(-1)^{\lvert w \rvert}$	Starts with a		Starts with b	
1	$(ab)^n$	$(+,+)$	$(ba)^n$	$(+,-)$
-1	$(ab)^n a$	$(-,+)$	$(ba)^n b$	$(-,-)$

Figure 1: The traces of (non-degernate) alternating words

The non-degenerate alternating words are partitioned into the families $(\pm,\pm)$ as specified in Figure 1. The **trace** of a word is its family, denoted by $\mathrm{Tr}(w)$. For example, $\mathrm{Tr}(aba) = (-,+)$.

We extend the preceeding definitions from alternating words to all words as follows. Say that w is **non-degenerate** whenever $\overline{w}$ is non-degenerate. Observe that w is non-degenerate iff it contains the substring aba. For any non-degenerate word w, define $\mathrm{Tr}(w) = \mathrm{Tr}(\overline{w})$.

We define trace concatenation in the unique manner such that

$$\mathrm{Tr}\, wx = \mathrm{Tr}\, w\, \mathrm{Tr}\, x.$$

2.4 Elements of arm classes

Introduce the set $\mathcal{A}$ of all elements of arm classes,

$$\mathcal{A} = \bigcup_{w \in \mathcal{M}} w.$$

Thus $\mathcal{A}$ consists of the arms which are concatenations of the maximal arms $(* - x)$ and $(*/x)$.

Theorem 2.14. *Every arm* $\mu \in \mathcal{A}$ *is connected.*

Of course if we were working in $\mathbb{R}$, this result would be immediate (since intersections of connected sets are connected). On $\mathbb{S}^1$ this is no longer true: for instance, consider the intersection of circular intervals

$$(0,\infty) \cap (2,1) = (2,\infty) \cup (0,1).$$

The trouble only occurs when the intervals wrap around ∞.

Claim 2.15. *If μ, ν are connected arms such that $\infty \notin \mathrm{Dom}(\mu) \cup \mu\,\mathrm{Dom}(\nu)$, then $\mu \rtimes \nu$ is connected.*

Proof. Since all arm functions are homeomorphisms, $\mu\,\mathrm{Dom}(\nu)$ is connected. By hypothesis, each of the sets $\mathrm{Dom}(\mu)$ and $\mu\,\mathrm{Dom}(\nu)$ are contained in $\mathbb{R}$. Therefore

$$\mathrm{Dom}(\mu \rtimes \nu) = \mathrm{Dom}(\mu) \cap \mu\,\mathrm{Dom}(\nu)$$

is an intersection of connected subsets of $\mathbb{R}$ and therefore connected. □

We return to the theorem.

Proof of Theorem 2.14. Since $\mu \in \mathcal{A}$, we have $\mu \in w$ for some $w \in W$. Induct on $|w|$. The base case $|w| = 0$ is immediate, and for longer words we have $w = aw'$ or $w = bw'$ for some word $w' \in \mathcal{M}$. Hence we may write $\mu = (h) \rtimes \nu$ for $h = (* - x)$ or $(*/x)$, and $\nu \in w'$. By the inductive hypothesis ν is connected. Since $\mathrm{Dom}(\nu) \subset \mathbb{R}^+$, it follows that $\infty \notin \mu\,\mathrm{Dom}(\nu)$. Lastly, note that (h) is connected and $\infty \notin \mathrm{Dom}((h))$. Hence by Claim 2.15, μ is connected. □

There is another arm property that will arise in the factorization problem.

Definition 2.16. *Let f be a homeomorphism of $\overline{\mathbb{R}}$. We say that f **factorizable** if*

$$f(0), f(\infty), f^{-1}(0), f^{-1}(\infty) \in \mathbb{R}^+.$$

Note: If any 3 of these quantities lies in $\mathbb{R}^+$, so does the 4th.

Example 2.17. *The function $f(x) = 1 - \frac{2}{1-x}$ is not factorizable since $f(0) = -1$.*

Observe that $(f, \varnothing) \in aba$. In general if $\mu \in w$ has a non-factorizable function, then μ is empty.

Suppose that f is factorizable and $\deg f = 1$. Then we compute $\mathbb{R}^+ \cap f^{-1}\mathbb{R}^+$ as follows:

$$(0,\infty) \cap f^{-1}(0,\infty) = (0,\infty) \cap (f^{-1}0, f^{-1}\infty) = (0, f^{-1}\infty) \cup (f^{-1}0, \infty).$$

Similarly when $\deg f = -1$, we have

$$\mathbb{R}^+ \cap f^{-1}\mathbb{R}^+ = (0, f^{-1}0) \cup (f^{-1}\infty, \infty).$$

Hence the maximal arm (f) has two components. We name them $C_{\pm 1}$ as follows:

$$0 < C_+ < C_- < \infty.$$

Example 2.18. *The function $f(x) = 2 - \frac{1}{1-x}$ is factorizable. The maximal arm (f) has domain*

$$(0, \frac{1}{2}) \cup (1, \infty), \qquad C_+ = (0, \frac{1}{2}) \text{ and } C_- = (1, \infty).$$

Definition 2.19. *Let $\mu = (f, U)$ be a non-empty connected factorizable arm. The **trace** of μ is:*

$$\mathrm{Tr}(\mu) = (\deg f, \textit{index of component containing } U).$$

Example 2.20. *If $\mu = \left(2 - \frac{1}{1-x}, (0, \frac{1}{2})\right)$, then $\mathrm{Tr}(\mu) = (-1, 1)$. Observe that $\mu \in aba$.*

2.5 The factorization problem

Given an arm μ and a word w, we would like to determine if μ can be written as the restriction of an element of w. This is equivalent to finding coefficients which "factor" the arm as an appropriate concatenation of arms in a, b, while ensuring positivity. When w is an alternating word, we can obtain a complete description of the elements of w.

For example, start by considering a maximal arm (f) with $0 < f^{-1}\infty < f^{-1}0 < \infty$ and $\deg f = 1$. We would like a representation of f as an element of an alternating word on a suitable domain. Since all elements of $a \cup b$ are orientation reversing, it is clear that the word has even length. Observe that

$$\mathrm{Dom}((f)) = (0,\infty) \cap f^{-1}(0,\infty) = (0,\infty) \cap (f^{-1}0, f^{-1}\infty) = (0, f^{-1}\infty) \cup (f^{-1}0, \infty).$$

Hence (f) has disconnected domain, so by Theorem 2.14 it does not arise from any arm class. However, restricting its domain to either component yields elements which do arise from words. Specifically, f can be produced by the non-degenerate words with trace $(+,+)$ and $(+,-)$.

In fact, we can show that the functions which appear in alternating arm classes are precisely those Möbius functions with maximal domain

$$(0, f^{-1}\infty) \cup (f^{-1}0, \infty), \qquad (0, f^{-1}0) \cup (f^{-1}\infty, \infty)$$

in the even and odd parity cases, respectively.

A little casework establishes the relative locations of the root and singularity of f, which depends on $\deg f$. In each case, the two conditionals we have presented are equivalent.

$\deg f$	Values of f	Values of f^{-1}
1	$[[0, f\infty, f0, \infty]]$	$[[0, f^{-1}\infty, f^{-1}0, \infty]]$
-1	$[[0, f0, f\infty, \infty]]$	$[[0, f^{-1}0, f^{-1}\infty, \infty]]$

Figure 2: Structure of factorizable functions

w	Any non-empty $\mu \in w$ has this structure:						$\mathrm{Tr}(w)$
$(ab)^n$	$(0$	$\overset{\mu\infty}{(ab)^n\infty)}$	$\overset{\mu0}{(ab)^{n-1}\infty}$	$\cdots$	$(ab)\infty$	∞	$(+,+)$
$(ab)^n a$	$(0$	$\overset{\mu0}{(ab)^{n+1}\infty)}$	$\overset{\mu\infty}{(ab)^n\infty}$	$\cdots$	$(ab)\infty$	∞	$(-,+)$
$(ba)^n$	0	$(ba)0$	$\cdots$	$\overset{\mu\infty}{(ba)^{n-1}0}$	$\overset{\mu0}{((ba)^n0}$	$\infty)$	$(+,-)$
$(ba)^n b$	0	$(ba)0$	$\cdots$	$\overset{\mu0}{(ba)^{n-1}0}$	$\overset{\mu\infty}{((ba)^n0}$	$\infty)$	$(-,-)$

Figure 3: Structure of alternating arms

2.6 Existence of arm factorizations

The main theorem of this section is that arm factorizations exist in great generality. Before proving the theorem, we introduce a slightly technical pair of lemmas.

An arm is FRC if it is Möbius, non-empty, regular, connected, and its function is factorizable.

Lemma 2.21. *Consider an FRC arm μ. If $\nu \in b$, then $\mu \rtimes \nu$ and $\nu \rtimes \mu$ are FRC and*

$$\operatorname{Tr}\mu \rtimes \nu = \operatorname{Tr}\mu \operatorname{Tr}\nu, \qquad \operatorname{Tr}\nu \rtimes \mu = \operatorname{Tr}\nu \operatorname{Tr}\mu.$$

Proof. It is clear that the arm is Möbius. Introducing ν doesn't affect the intervals, since $\nu\mathbb{R}^+ = \mathbb{R}^+$. Hence it is still non-empty, regular, and connected. Lastly, observe that factorizability is preserved since introducing ν simply permutes $0, \infty$.

To check the trace condition, observe that the $\deg f$ statement is immediate. For the component condition, check both cases separately. □

Lemma 2.22. *Consider an FRC arm μ such that $\operatorname{Dom}\mu \subset (0, y)$, and let $\nu = (y - x) \in a$ denote a maximal arm. Then $\mu \rtimes \nu$ and $\nu \rtimes \mu$ are FRC and*

$$\operatorname{Tr}\mu \rtimes \nu = \operatorname{Tr}\mu \operatorname{Tr}\nu, \qquad \operatorname{Tr}\nu \rtimes \mu = \operatorname{Tr}\nu \operatorname{Tr}\mu.$$

Proof. The domain criteria ensure that introducing ν doesn't cause additional restrictions. Then the proof is analogous to Lemma 2.21. □

Theorem 2.23. *Let μ be an FRC arm, w a word that contains aba, and suppose $\operatorname{Tr}(\mu) = \operatorname{Tr}(w)$. Then there is a $\nu \in w$ such that $\mu \subset \nu$.*

Proof. Let $\mathrm{Thm}(w)$ denote the assertion with word w. We induct on w as follows.

Claim 2.24. $\mathrm{Thm}(aba)$ *holds.*

Proof. Let μ be an FRC arm with function f, such that $\operatorname{Tr}\mu = \operatorname{Tr} aba = (-, +)$. By Subsection 2.4,

$$0 < f^{-1}0 < f^{-1}\infty < \infty \implies 0 < f0 < f\infty < \infty,$$

and $\mu \subset \nu = (f, (0, f^{-1}0))$. Consequently we have the factorization

$$\nu = (f^{-1}\infty - x) \rtimes \left(f^{-1}\infty \frac{f\infty - f0}{x}\right) \rtimes (f\infty - x) \in aba.$$

Indeed, the domains on the left and right coincide by Subsection 2.4. Moreover the functions agree at $0, \infty, f^{-1}\infty$ by inspection. Since the group of Möbius functions is sharply 3-transitive, the functions coincide and thus we have equality. □

Claim 2.25. $\mathrm{Thm}(w) \implies \mathrm{Thm}(wb), \mathrm{Thm}(bw)$

Proof. Take μ FRC with $\operatorname{Tr}\mu = \operatorname{Tr} wb$ and choose any $\lambda \in b$. By Lemma 2.21, it follows that $\mu \rtimes \lambda$ is FRC and

$$\operatorname{Tr}(\mu \rtimes \lambda) = \operatorname{Tr} wb^2 = \operatorname{Tr} w.$$

Hence by $\mathrm{Thm}(w)$, there exists $\nu \in w$ such that $\mu \rtimes \lambda \subset \nu$. Consequently since $\lambda \rtimes \lambda = (\mathrm{id})$,

$$\mu = (\mu \rtimes \lambda) \rtimes \lambda \subset \nu \rtimes \lambda \in wb.$$

This establishes $\mathrm{Thm}(wb)$. An analogous manipulation establishes $\mathrm{Thm}(bw)$. □

Claim 2.26. $\mathrm{Thm}(w) \implies \mathrm{Thm}(wa), \mathrm{Thm}(aw)$

Proof. Since μ is regular, there exists a regular extension $\tilde{\mu}$ and $y \in \mathbb{R}^+$ such that $\mu \subset \tilde{\mu}$ and

$$\mu = \tilde{\mu} \rtimes (\mathrm{id}, (0, y)).$$

In fact since μ is FRC, we can arrange for $\tilde{\mu}$ to also be FRC. Let $\lambda = (y - x, (0, y)) \in a$. By Lemma 2.22, it follows that $\tilde{\mu} \rtimes \lambda$ is FRC. Proceeding just as in Claim 2.25, we observe that

$$\mu = (\tilde{\mu} \rtimes \lambda) \rtimes \lambda.$$

By $\mathrm{Thm}(w)$ we obtain $\nu \in W$ such that $\tilde{\mu} \rtimes \lambda \subset \nu$. Then we have $\mu \subset \nu \rtimes \lambda \in wa$, establishing $\mathrm{Thm}(wa)$. Analogously we obtain $\mathrm{Thm}(aw)$. □

Since all words w containing aba can be obtained by repeatedly appending a, b to the ends of aba, it follows by induction that $\mathrm{Thm}(w)$ holds for all w containing aba. □

Note that we have used several properties from Lemma 2.10 during the proof.