

# A 3-to-1 cactus graph: Details

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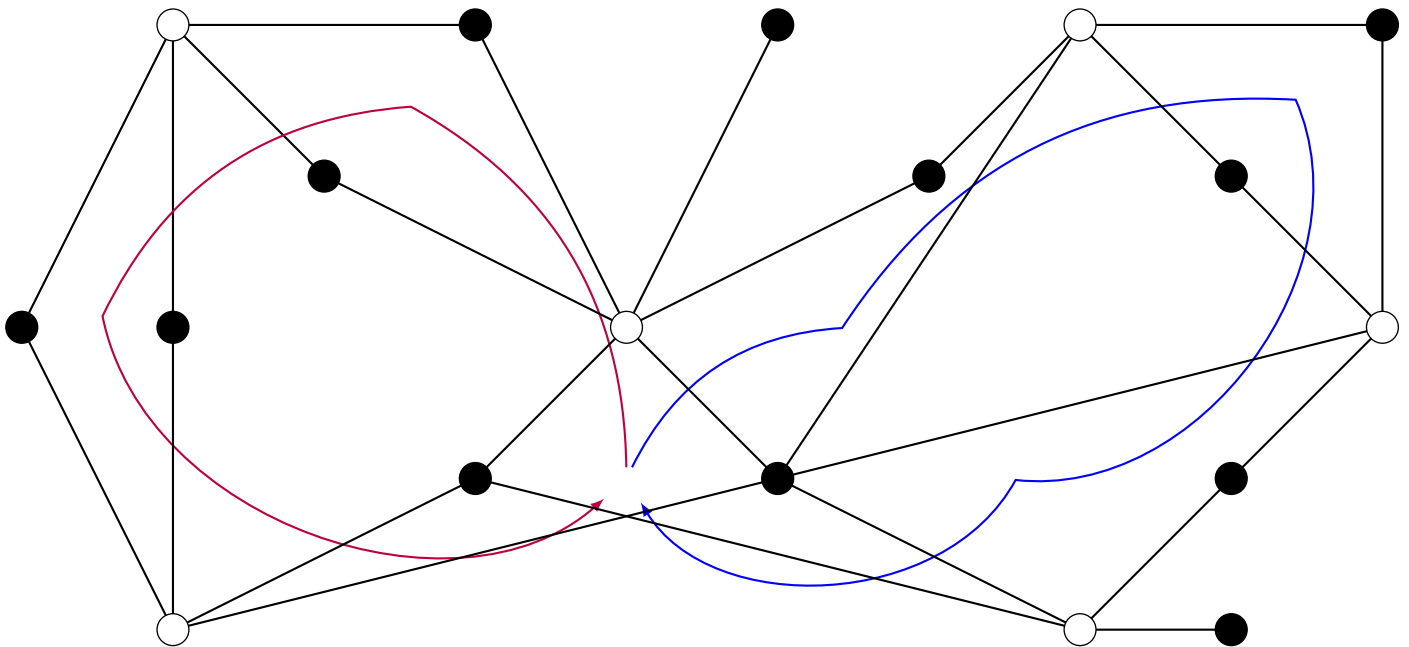


Figure 1: Overview of a two leaf cactus

The overview emphasizes the underlying loop structure. The left loop consists of  $\text{quad}^3$ , and the right consists of switch  $\text{quad}^2$  switch. Note that the overview omits some “non-structural” auxiliary edges which we introduce on the next page.

Notice that the loops emanate from the central 6-star, which is actually a (quad # switch). We also refer to it as a **multiplexor**, following [Klumb].

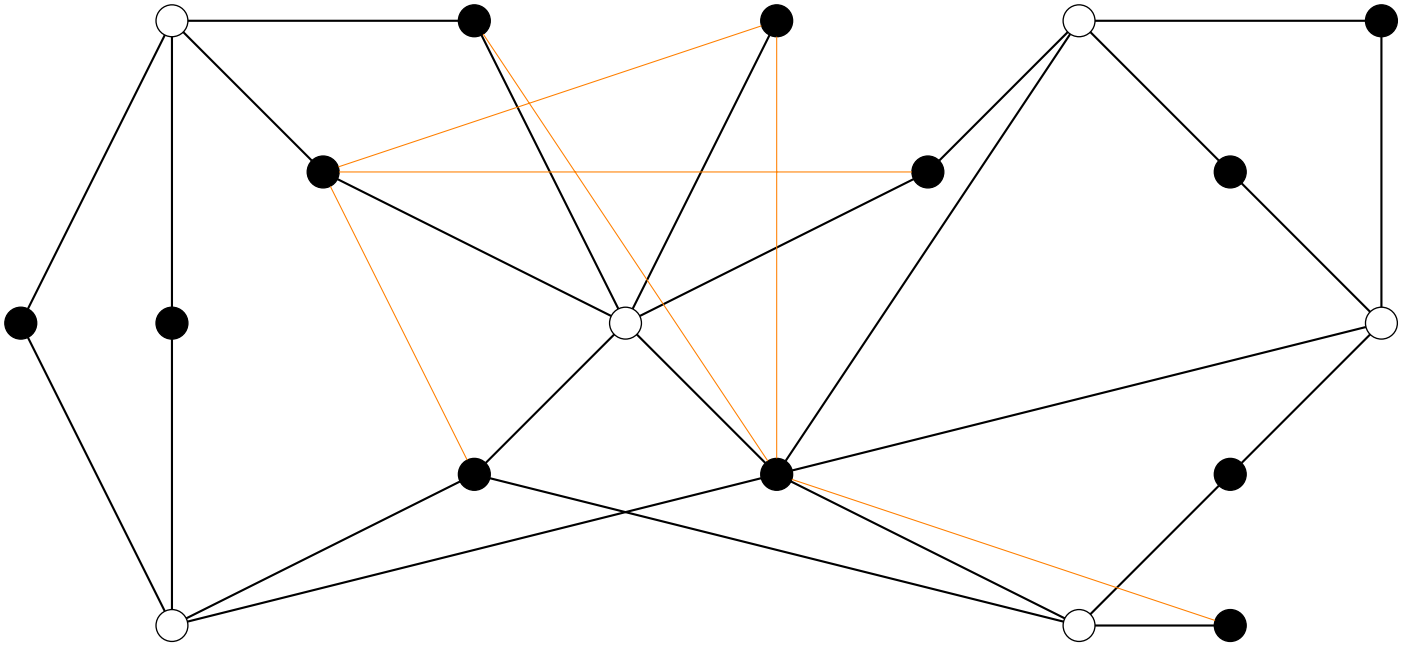


Figure 2: Auxiliary edges which prevent recoverability

Here  $\kappa(G)$  denotes the arity of  $G$  (cardinality of largest fiber).

**Claim 1 (Upper bound).**  $\kappa(G) \leq 3$

*Proof.* Let  $x$  denote the sum of the conductivities entering  $v_{13}v_{14}$  (see Figure 10). That is,  $x$  is the sum of the red and blue arrow heads.

Suppose  $x$  is known. By the claim following Figure 6, all edges in the multiplexor are determined. Propagation along the red and blue loops determines the remaining edges. Hence

$$\kappa(G) \leq \# \text{ of choices for } x.$$

Since there are 2 loops,  $x$  satisfies a cubic and thus the graph is at most 3-to-1. □

**Claim 2 (Lower bound).**  $\kappa(G) \geq 3$

*Proof.* We construct three networks on  $G$  with the same response. The networks correspond to  $x = 2, 3, 4$ . See tables on next page and subsequent diagrams for the rest of the construction. □

$$x \rightarrow 7-x \rightarrow \frac{1}{7-x} \rightarrow 2 - \frac{1}{7-x} \rightarrow \frac{6}{2 - \frac{1}{7-x}} \rightarrow 4 - \frac{6}{2 - \frac{1}{7-x}} \rightarrow \frac{1}{4 - \frac{6}{2 - \frac{1}{7-x}}} = 1 - \frac{3/2}{x-5}$$

2	5	$\frac{1}{5}$	$\frac{9}{5}$	$\frac{10}{3}$	$\frac{2}{3}$	$\frac{3}{2}$
3	4	$\frac{1}{4}$	$\frac{7}{4}$	$\frac{24}{7}$	$\frac{4}{7}$	$\frac{7}{4}$
4	3	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{18}{5}$	$\frac{2}{5}$	$\frac{5}{2}$

Figure 3: Arm propagation for **left loop** (quad<sup>3</sup>)

$$x \rightarrow 7-x \rightarrow 7-x \rightarrow x \rightarrow \frac{1}{x} \rightarrow 1 - \frac{1}{x} \rightarrow \frac{3/2}{1 - \frac{1}{x}} \rightarrow \frac{7}{2} - \frac{3/2}{1 - \frac{1}{x}} \rightarrow \frac{7}{2} - \frac{3/2}{1 - \frac{1}{x}} = 2 - \frac{3/2}{x-1}$$

2	5	5	2	$\frac{1}{2}$	$\frac{1}{2}$	3	$\frac{1}{2}$	$\frac{1}{2}$
3	4	4	3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{9}{4}$	$\frac{5}{4}$	$\frac{5}{4}$
4	3	3	4	$\frac{1}{4}$	$\frac{3}{4}$	2	$\frac{3}{2}$	$\frac{3}{2}$

Figure 4: Arm propagation for **right loop** (switch quad<sup>2</sup> switch)

Observe that all three assignments are valid due to “loop conservation”:

$$\begin{bmatrix} 3 \\ 2 \\ 7 \\ 4 \\ 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

In a sense the construction is complete, for we have found a valid loop assignment. However, it is instructive to illustrate the process of populating a graph given a loop assignment. Our present graph can be decomposed into quads, switches, and a (quad # switch), so it suffices to populate these three subgraphs. In the following diagrams, the expressions in the central interior node is a multiplier, to be applied to the surrounding edge weights.

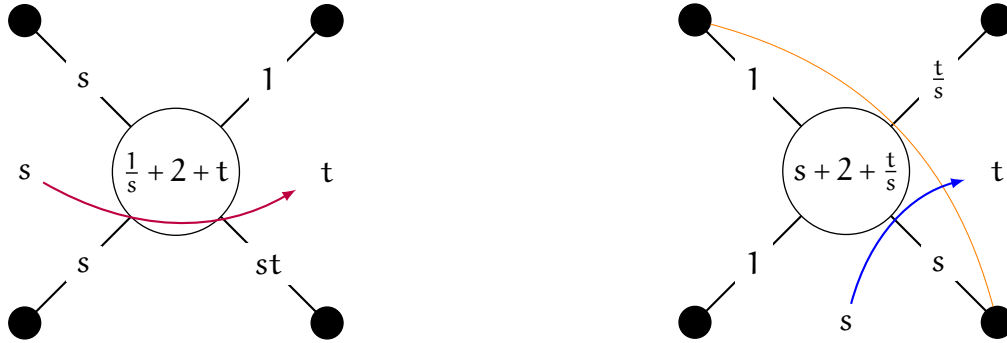


Figure 5: Populating a quad and switch, respectively.

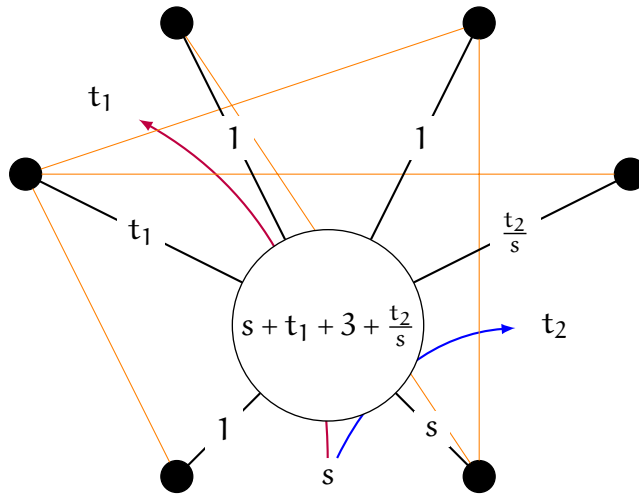


Figure 6: Populating a (quad # switch)

**Claim 3.** *The edge entering a (quad # switch) determines all edges.*

*Proof.* Play the following game: an orange edge is removed if its endpoints can be connected with white edges. Observe that all orange edges are removed. □



