A 3-to-1 cactus graph: Details

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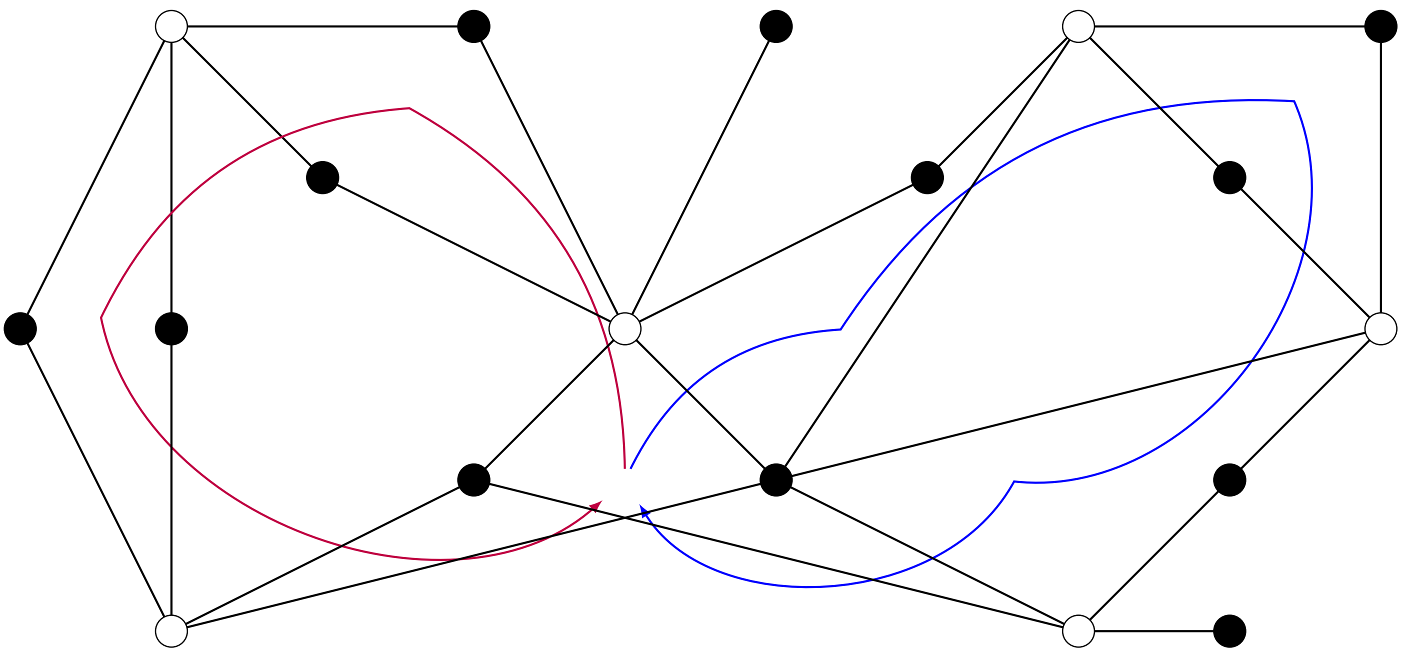

Figure 1: Overview of a two leaf cactus

The overview emphasizes the underlying loop structure. The left loop consists of quad^3, and the right consists of switch quad^2 switch. Note that the overview omits some "non-structural" auxiliary edges which we introduce on the next page.

Notice that the loops emanate from the central 6-star, which is actually a (quad # switch). We also refer to it as a **multiplexor**, following [Klumb].

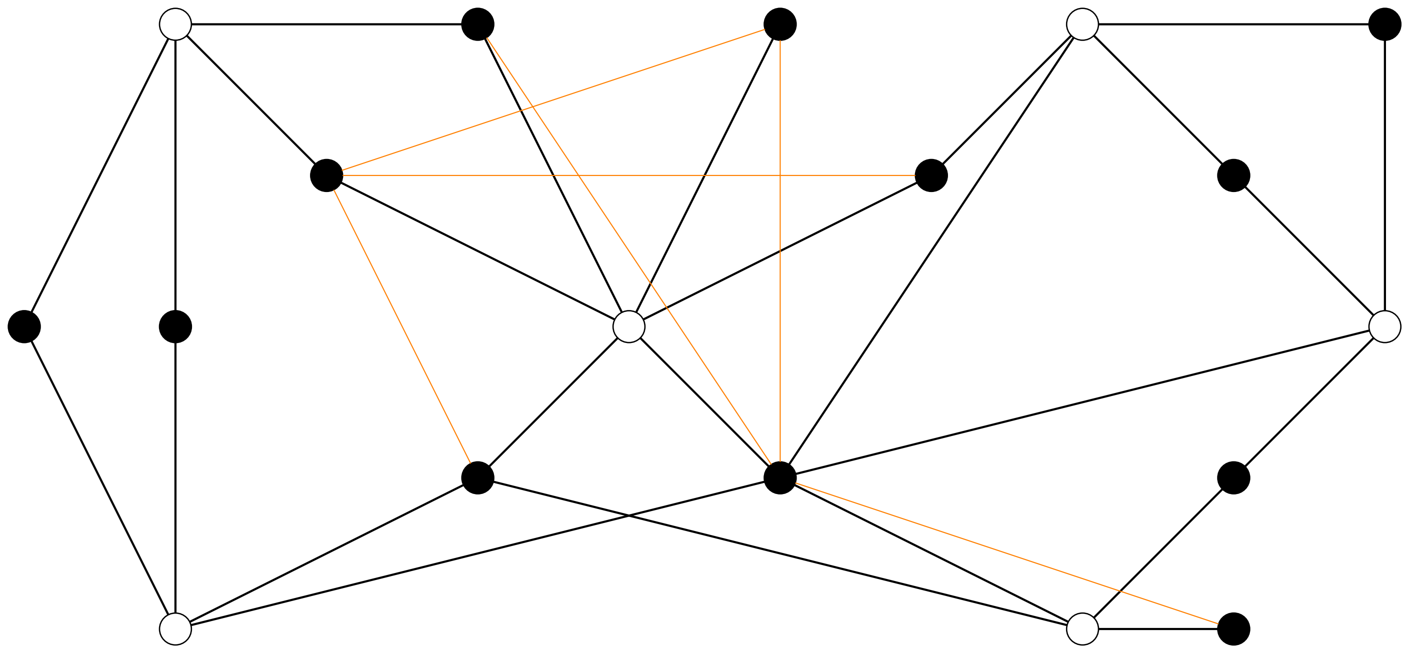

Figure 2: Auxiliary edges which prevent recoverability

Here $\kappa(G)$ denotes the arity of G (cardinality of largest fiber).

Claim 1 (Upper bound). $\kappa(G) \leq 3$

Proof. Let x denote the sum of the conductivities entering $v_{13}v_{14}$ (see Figure 10). That is, x is the sum of the red and blue arrow heads.

Suppose x is known. By the claim following Figure 6, all edges in the multiplexor are determined. Propagation along the red and blue loops determines the remaining edges. Hence

$$\kappa(G) \leq \#\text{ of choices for } x.$$

Since there are 2 loops, x satisfies a cubic and thus the graph is at most 3-to-1. □

Claim 2 (Lower bound). $\kappa(G) \geq 3$

Proof. We construct three networks on G with the same response. The networks correspond to $x = 2, 3, 4$. See tables on next page and subsequent diagrams for the rest of the construction. □

$$x \longrightarrow 7-x \longrightarrow \tfrac{1}{7-x} \longrightarrow 2-\tfrac{1}{7-x} \longrightarrow \tfrac{6}{2-\frac{1}{7-x}} \longrightarrow 4-\tfrac{6}{2-\frac{1}{7-x}} \longrightarrow \tfrac{1}{4-\frac{6}{2-\frac{1}{7-x}}} = 1-\tfrac{3/2}{x-5}$$

2	5	$\frac{1}{5}$	$\frac{9}{5}$	$\frac{10}{3}$	$\frac{2}{3}$	$\frac{3}{2}$
3	4	$\frac{1}{4}$	$\frac{7}{4}$	$\frac{24}{7}$	$\frac{4}{7}$	$\frac{7}{4}$
4	3	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{18}{5}$	$\frac{2}{5}$	$\frac{5}{2}$

Figure 3: Arm propagation for left loop (quad^3)

$$x \longrightarrow 7-x \longrightarrow 7-x \longrightarrow x \longrightarrow \tfrac{1}{x} \longrightarrow 1-\tfrac{1}{x} \longrightarrow \tfrac{3/2}{1-\frac{1}{x}} \longrightarrow \tfrac{7}{2}-\tfrac{3/2}{1-\frac{1}{x}} \longrightarrow \tfrac{7}{2}-\tfrac{3/2}{1-\frac{1}{x}} = 2-\tfrac{3/2}{x-1}$$

2	5	5	2	$\frac{1}{2}$	$\frac{1}{2}$	3	$\frac{1}{2}$	$\frac{1}{2}$
3	4	4	3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{9}{4}$	$\frac{5}{4}$	$\frac{5}{4}$
4	3	3	4	$\frac{1}{4}$	$\frac{3}{4}$	2	$\frac{3}{2}$	$\frac{3}{2}$

Figure 4: Arm propagation for right loop (switch quad^2 switch)

Observe that all three assignments are valid due to "loop conservation":

$$\begin{bmatrix} \frac{3}{2} \\ \frac{7}{4} \\ \frac{5}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{5}{4} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

In a sense the construction is complete, for we have found a valid loop assignment. However, it is instructive to illustrate the process of populating a graph given a loop assignment. Our present graph can be decomposed into quads, switches, and a (quad # switch), so it suffices to populate these three subgraphs. In the following diagrams, the expressions in the central interior node is a multiplier, to be applied to the surrounding edge weights.

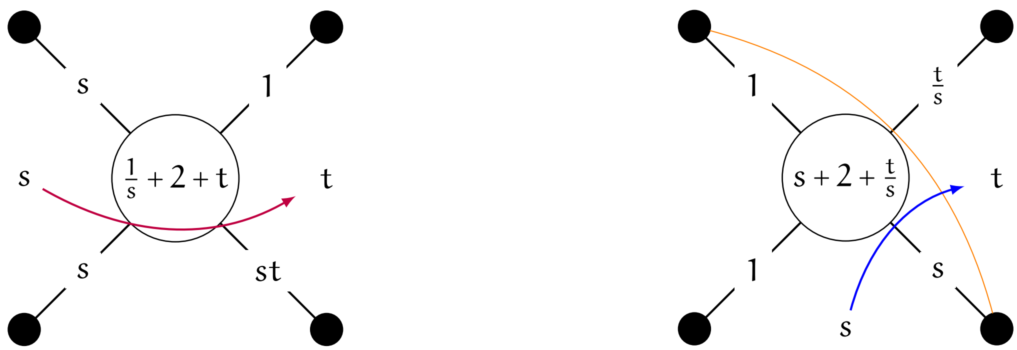

Figure 5: Populating a quad and switch, respectively.

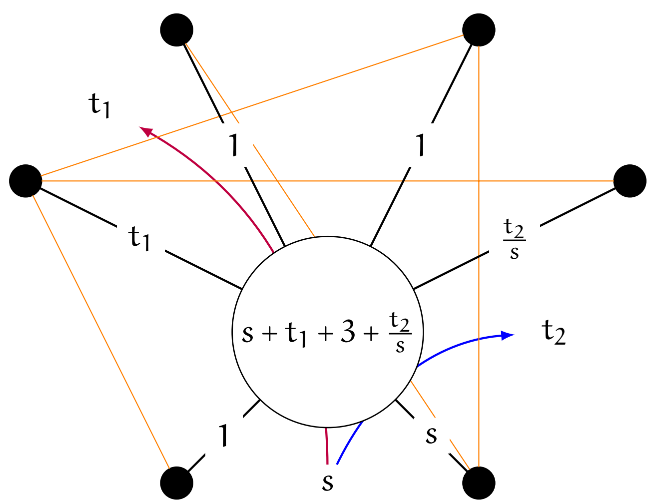

Figure 6: Populating a (quad # switch)

Claim 3. *The edge entering a (quad # switch) determines all edges.*

Proof. Play the following game: an orange edge is removed if its endpoints can be connected with white edges. Observe that all orange edges are removed. $\square$

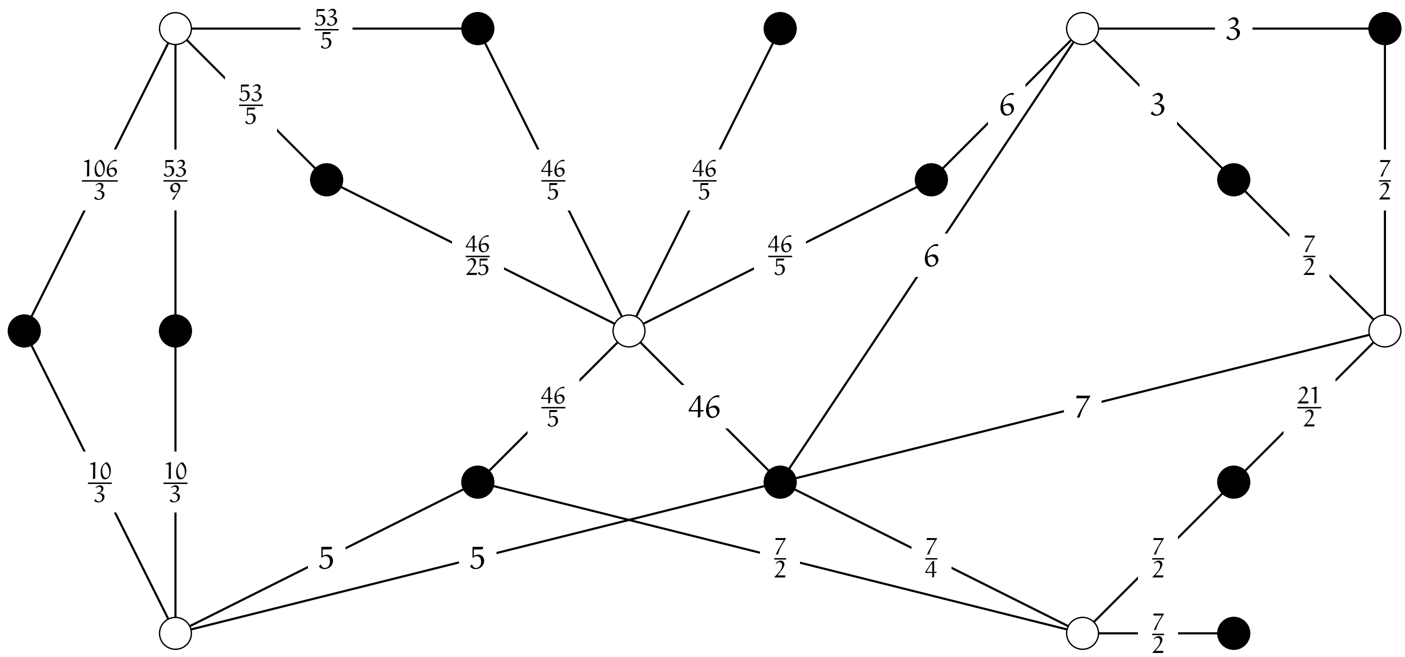

Figure 7: Populated with $x = 2$

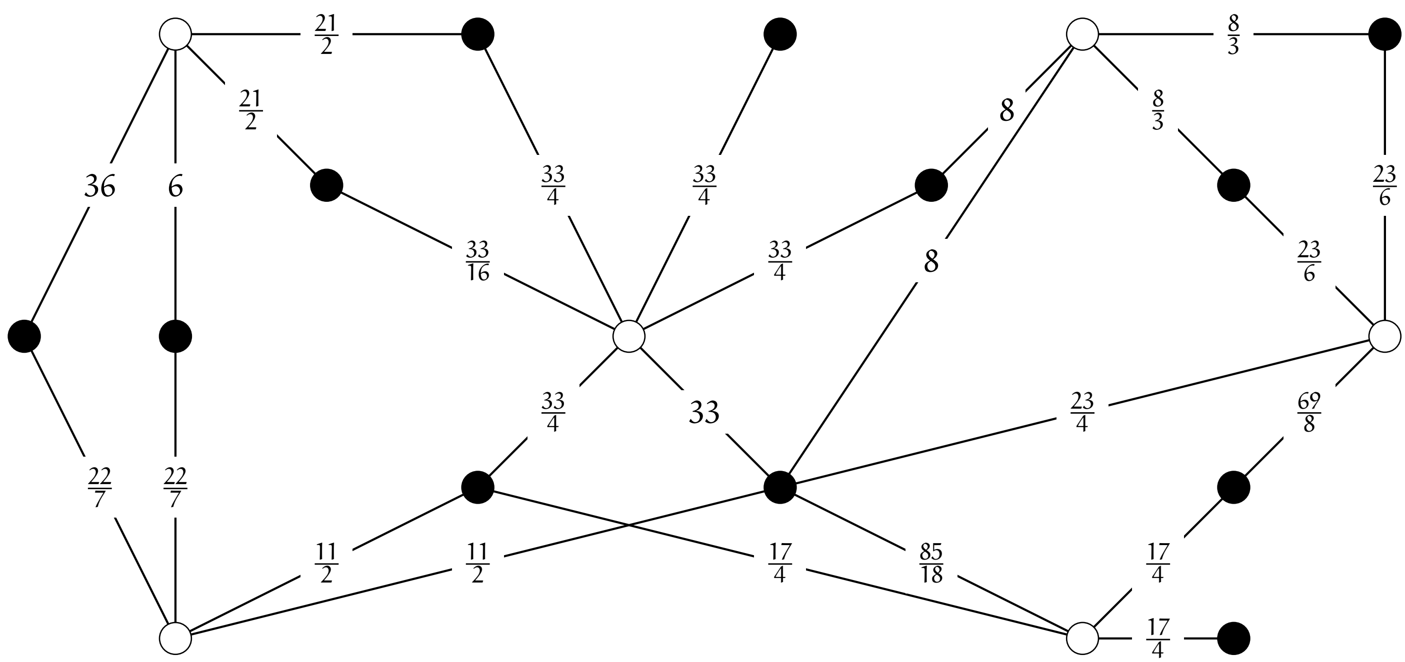

Figure 8: Populated with $x = 3$

Note that we didn't label the auxiliary edges, because it is clear that auxiliary edges can be chosen arbitrarily to ensure proper conductivities.

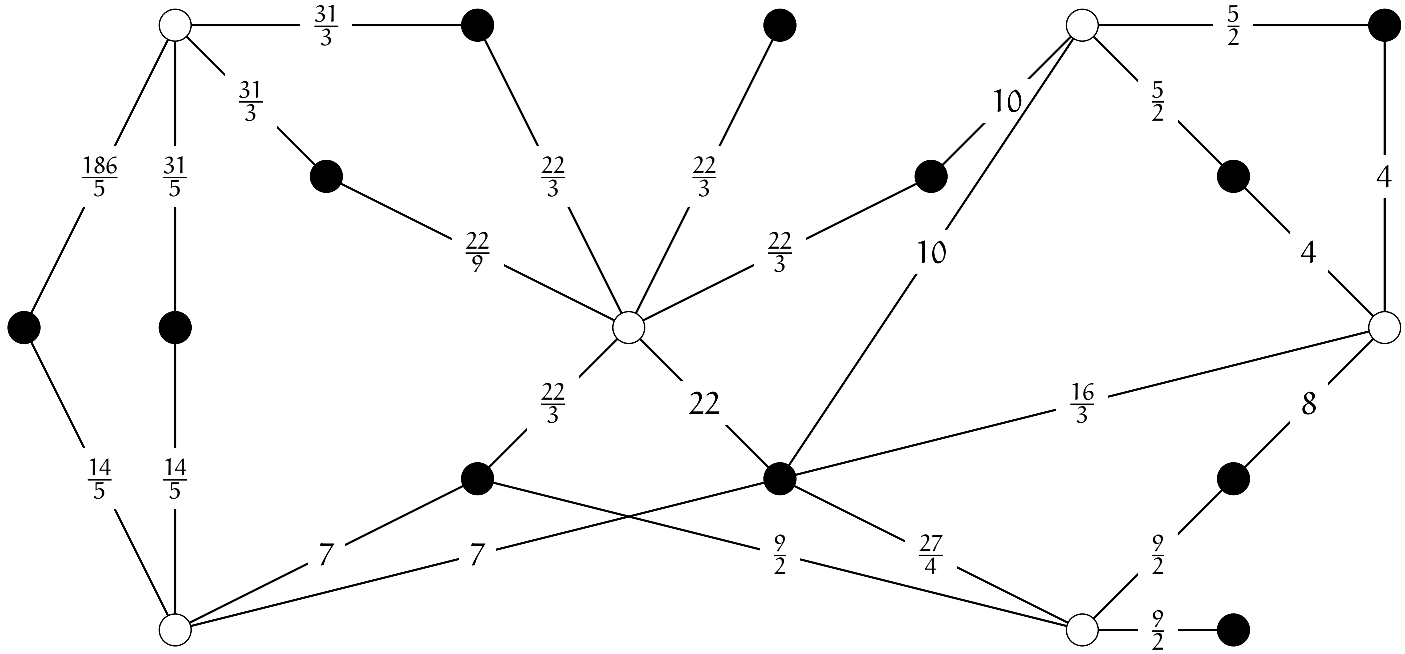

Figure 9: Populated with $x = 4$

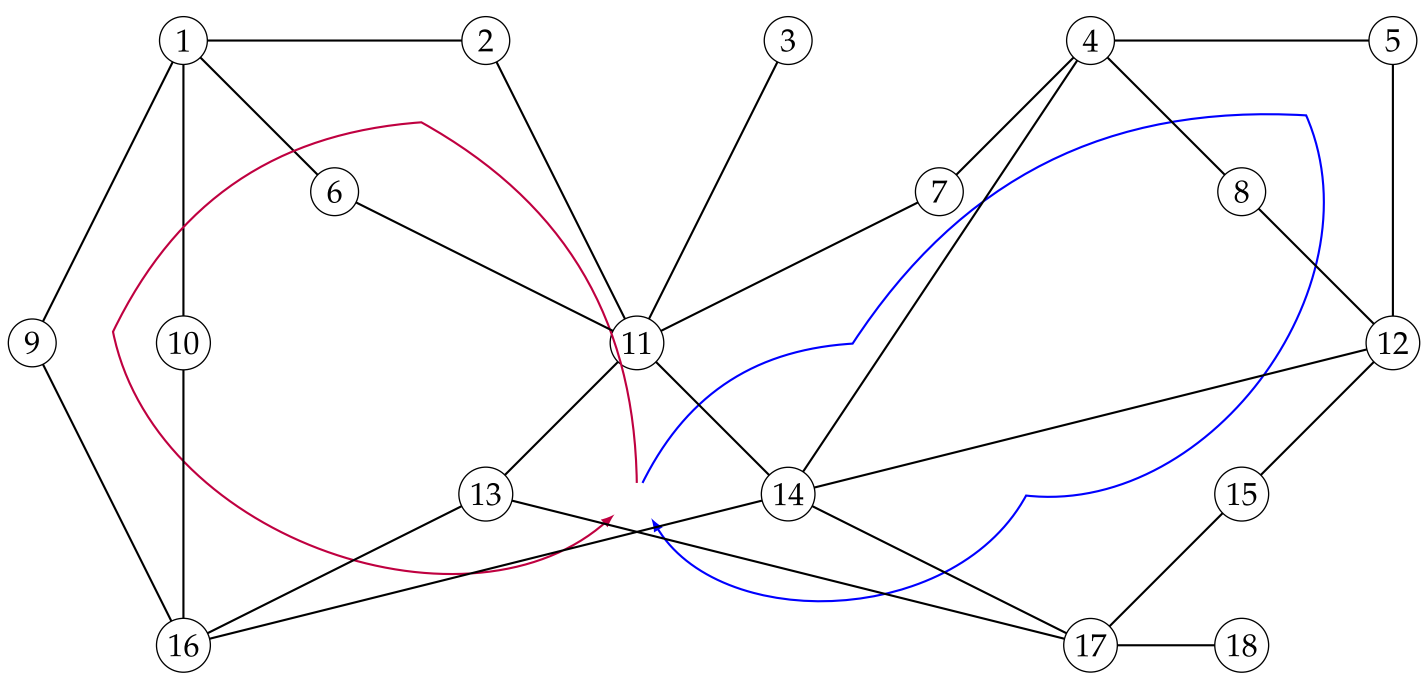

Figure 10: Standard vertex labelling