On the 544 final exam, several people made statements suggesting that to prove a function on a product space is continuous, it suffices to check that it is continuous in each variable separately. *This is not the case.* Part (a) of the problem below is intended to emphasize this fact. Part (b) is intended to prevent a somewhat similar misconception about differentiability.

Reminder of some definitions. A function f on an open subset of a product space $X \times Y$ is said to be *continuous separately on* X or *continuous separately in* x if for each $y_0 \in Y$, the function $x \mapsto f(x, y_0)$ is continous on X wherever defined. Separate continuity on Y or in y is defined simiarly. The definitions for differentiability and directional derivatives of a function on $\mathbb{R}^n$ are given on p. 642 and p. 647 of ISM.

Problem 0-1. Give an example of each of the following. (This of course means prove they are examples. You may find some of the theorems in ISM Appendix C useful in writing the proofs.)

(a) A function $f(x, y)$ on an open set in $\mathbb{R}^2$ that is continuous in each variable separately but is not continuous on its domain.

(b) A function $f(x, y)$ on an open set in $\mathbb{R}^2$ that has a directional derivative in every direction at every point of its domain, but is not differentiable everywhere on its domain.

Remark, not part of the problem. Note that partial derivatives are special cases of directional derivatives. Thus part (b) shows that differentiability is a stronger condition than the existence of partial derivatives.

Not part of the homework. You also should know of the existence of a function whose mixed second partials are not equal.