The term "regular" is used in mathematics in many different ways depending on the context. Two kinds of *regular points*, which are related but not the same, occur in the study of implicit functions.

The first kind we will consider is a *regular point for a function.* Recall that a *critical point for a function* $\mathbb{R}^n \to R$ is a point at which the total derivative at the point is zero; that is, all the partials vanish. All the other points in the domain of the function are called *regular points* for the function. In other words, a *regular point for a function* $\mathbb{R}^n \to R$ is a point at which at least one of the partials is not zero. By adding the condition that we are at a regular point, we can obtain a partial converse to the Implicit Function Theorem in §§8.1 and 8.2.

Proposition 1: Addition to the Implicit Function Theorem.
Notation: Let $\boldsymbol{x}$ be a point in $\mathbb{R}^n$ and $z \in \mathbb{R}$; thus $(\boldsymbol{x}, z)$ is a point in $\mathbb{R}^{n+1}$. Suppose $F(\boldsymbol{x}, z)$ is a function $F : \mathbb{R}^{n+1} \to R$ with continuous first partial derivatives on a neighborhood of a point P_0 that is a regular point for F. Assume that $F(P_0) = 0$ and $\partial F/\partial z(P_0) = 0$. Then it is *not* possible to find a *differentiable* function $f(\boldsymbol{x})$ so that

$$F(\boldsymbol{x}, z) = 0 \text{ is equivalent to } z = f(\boldsymbol{x})$$

on any neighborhood of P_0.

Proof. We will assume that there is a differentiable function $z = f(\boldsymbol{x})$ such that $F(\boldsymbol{x}, f(\boldsymbol{x})) = 0$, and show this leads to a contradiction. That will mean our assumption is false, so there is no such function f.

Because P_0 is a regular point for F, (at least) one of the first partials of F at P_0 is not zero: say it's the kth one, so

$$\frac{\partial F}{\partial x^k}(P_0) \neq 0.$$

Differentiate $F(\boldsymbol{x}, f(\boldsymbol{x})) = 0$ with respect to x^k to get

$$\frac{\partial F}{\partial x^k}(P_0) + \frac{\partial F}{\partial z}(P_0)\frac{\partial f}{\partial x^k}(P_0) = 0.$$

Putting all the information together, we have

$$0 \neq \frac{\partial F}{\partial x^k}(P_0) = -\frac{\partial F}{\partial z}(P_0)\frac{\partial f}{\partial x^k}(P_0) = 0\left(\frac{\partial f}{\partial x^k}(P_0)\right) = 0.$$

This contradiction proves our result. ■

Slight generalization: If $F(P_0) = c \neq 0$, change the equation in the conclusion to $F(\boldsymbol{x}, z) = c$.

Questions still not answered. If we have the conditions in the proposition and $\partial F/\partial z(P_0) = 0$, we know there is no *differentiable* function f so $F(\boldsymbol{x}, z) = 0$ is equivalent locally to $z = f(\boldsymbol{x})$; but there might still be a continuous one. (Example: $F(x, y, z) = x + y - z^3$ at the origin.) And at critical points of F, we have no conclusion from the theorem in the book or Proposition 1. To find out more about the these issues, we just have to investigate the particular equation given. Things to try: Can you actually solve the equation for z (or whichever variable you are asked about)? If so, is the result single-valued? defined for all nearby values of the other variables x and y? differentiable with respect? Is x or y at its maximum or minimum possible value for points on the locus, so that we can't define a function of x and y on a neighborhood of the point?

Homework hints. In exercises 8.2.3 and 8.2.20, you are asked if you can find one of the variables in terms of the others. First use Prop. 1 to determine whether you can find it as a *differentiable* function of the others. Then use the suggestions above to try to answer the harder question about whether it can be given as just a *continuous* funciton of the others. This is a little tricky in 8.2.3, but in 8.2.20 the second part of the problem gives you a picture that will help you answer the question.

The second kind of regular point is a *regular point for a locus.* (Recall locus just means the solution set for an equation. Sometimes we will say surface instead of locus.) This is defined to be a point on the locus where, locally, the locus is exactly equal to the graph of some differentiable function. As usual, "locally" means on some neighborhood of the point.

(In the book, a slightly different definition is given in Exercise 8.2.7: the function is not required to be differentiable, only continuous. I'm adding the condition that the function be differentiable because the graph of a continuous function can still have corners and cusps, which is not very "regular.")

Although the two meanings of "regular point" are not the same, the Implicit Function Theorem gives us a relation between them. For simplicity, the following result is stated only for functions of three variables, which is the case we will use. The generalization to functions of more variables also holds, with essentially the same proof.

Proposition 2: Exercise 8.2.7(a)).
Suppose $P_0 = (x_0, y_0, z_0)$ is a regular point of the function $F(x, y, z)$ which has continuous first partial derivatives on a neighborhood of P_0. Then P_0 is a regular point of the locus $F(x, y, z) = F(x_0, y_0, z_0)$.

Proof. Because P_0 is a regular point of F, at least one of the first partials of F is nonzero at P_0. Then the Implicit Function Theorem (either as stated in the book, or in Prop. 1 above) tells us that on some neighborhood of P_0, the locus is given as the graph of a continuous function. (The theorems are stated for the case when it's the partial with respect to z that is nonzero, but by interchanging z with x or with y, we get the analogous result for the case when a different partial is nonzero.) ■

Problem G. This is a modification of Exercise 8.32.7(bc).
(b) Consider the cone locus $x^2 + y^2 = z^2$. Which points on this locus are regular points of the locus? At the singular point(s) of the locus, show that you can't find any one of the variables as a function of the other two, no matter how small a neighborhood of the point you consider. (Hint: See the paragraph *Questions still not answered* above.)
(c) For $a \neq 0$, let $F(x, y, z) = (x^2 + y^2 + z^2)^2 - a^2(x^2 + y^2 + z^2)$. Describe the locus $F(x, y, z) = 0$. (Hint: it has two pieces.) Which points of the locus are regular points for F? For the locus?

Extending these ideas to vector valued functions. If the function F is vector-valued, a regular point is one where the total derivative (matrix) has linearly independent rows. For instance, if we have a continouous and continuously differentiable function $F(x, y, z, u, v) = (r, s)$, the total derivative is a two by five matrix. The rows will be linearly independent if there is a two by two submatrix with nonzero determinant; that is, at least one of the Jacobian determinants

$$\frac{\partial(r,s)}{\partial(x,y)} \text{ or } \frac{\partial(r,s)}{\partial(x,u)} \text{ or } \frac{\partial(r,s)}{\partial(z,v)} \text{ or ... (ten different possibilities total up to } \pm)$$

must be nonzero. (Recall your linear algebra: why is this the case?) Then the (generalized) Implicit Function Theorem of §8.3 tells us the corresponding two independent variables (x, y, or x, u, or z, v, respectively, for the three determinants written above) can be solved for as differentiable functions of the other three.

A locus given by one equation in more than three variables with every point a regular point is called a *regular hypersurface*. (My alter ego, the "exuberant mathematician" who likes to tell you more advanced ideas, can't resist telling you this can generalized still further to a concept called a *smooth manifold*. If this interests you, consider taking Math 441/2/3.)