

This series of brainteasers previews important ideas in Chapter 9, the chapter we'll study after chapter 6 for our final work of the quarter. We'll discuss parts (a) and (b) on Monday, Nov. 27, and the other parts later in the week.

Introduction. The HOTEL INFINITY (also known as “Hilton’s Grand Hotel,” after the mathematician David Hilbert) has an infinite number of rooms, identified by the room number. For every natural number, there is one and only one room with that number. Each room will accommodate only one guest. On Sunday night, every room is full, and the guests are all planning to stay all week, through Friday night.

(a) On Monday morning, the Lonely Mathematician, or “LM” for short, arrives and asks for a room for Monday through Friday nights. When told all the rooms are full, LM starts to leave. “Don’t worry,” says the clerk, “I can move the guests around so that everyone already here still has a room, and there will be one for you, too.” How can the clerk do this?

(b) LM is impressed by the clerk’s solution. On Monday afternoon, LM decides to interpret this solution as a function denoted f . The elements of domain of the function will be numbers representing the guests. For the previous guests, the number will be the number of the room they stayed in on Sunday night. “And I will be represented by the number zero,” thinks LM. The codomain of the function will be the number of the new room the guest is assigned to, so is the natural numbers \mathbb{N} . What is the formula and domain for f ?

Note that the function must be injective, because only one person may be assigned to each room.

(c) On Tuesday morning, LM is sitting in the hotel lobby when five new guests arrive and ask for rooms for the rest of the week. At first, the clerk proposes a plan which will make the current guests move 5 times. But LM steps up and says there is a way to make room for the new guests and ask each of the current guests to move only once. To do this, LM constructs a function g to assign rooms for these 5 new guests and all the current guests (including LM). What is g and what is its domain?

(d) On Wednesday morning, LM is again in the lobby when new guests arrive, but this time it is a large party of gregarious mathematicians. In fact there are an infinite number of them, and each of them has been assigned a distinct negative integer as a badge number. (“Positive integers are so last year,” remarks one of the newcomers.) This time the clerk is completely at a loss to figure out how to assign rooms for all the current current guests and also the new ones. “I certainly can’t ask the current guests to move an infinite number of times! It’s so inconvenient, even if we can get them to move fast enough that it doesn’t take an infinite amount of time.” But LM has a solution in the form of a function h . “And,” says LM, “my solution is a function that is not only injective, it’s bijective!” All the incoming guests exclaim in delight.

The codomain of h is still the natural numbers \mathbb{N} . Using the current room numbers for the current guests and the negative badge numbers for the newly arrived gregarious mathematicians, the domain for h is \mathbb{Z} , the integers. Define a bijective function h that LM can propose to the clerk.