Research Statement

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I am generally interested in algebraic number theory, algebraic groups and arithmetic groups. My dissertation research concerns the structure of certain types of S-arithmetic groups. Specifically, I have demonstrated that two families of groups have bounded generation.

1 Bounded Generation

Definition. A group G is said to have **bounded generation** if there exist elements $g_1, \ldots, g_n \in G$ such that $G = \langle g_1 \rangle \langle g_2 \rangle \cdots \langle g_n \rangle$, where $\langle g_i \rangle$ denotes the subgroup generated by g_i.

Any finite group trivially has bounded generation. We can also see that

$$D_\infty = \{a, b \mid a^2 = b^2 = 1\}$$

has bounded generation since $D_\infty = \langle a \rangle \langle ab \rangle \langle b \rangle$, in fact any polycyclic group has bounded generation. It can also be shown that F_n, the free group on n generators, does not have bounded generation for any $n \geq 2$.

It is well known that $SL_n(\mathbb{Z})$ is generated by elementary matrices. In [2] Carter and Keller establish that $SL_n(\mathbb{Z})$ has bounded generation with respect to elementary matrices, if $n \geq 3$. More generally, $SL_n(\mathcal{O})$ has bounded generation for the ring of integers $\mathcal{O}$ of a number field and $n \geq 3$.

This property is particularly useful when considering S-arithmetic groups. Let K be an algebraic number field, and let S be a finite set of valuations including all archimedean ones. We will denote by $\mathcal{O}_S$ the ring of S-integers, i.e., $\{x \in K \mid |x|_v \leq 1 \text{ for all valuations } v \notin S\}$

It was established by Tavgen in [6] that elementary subgroups of certain Chevalley groups over rings of S-integers have bounded generation. Rapinchuk and Erovenko showed in [3] that $\mathrm{Spin}(V)_{\mathcal{O}_S}$ (and thus the orthogonal group $O(V)$) has bounded generation whenever either V has Witt index at least 2, or S contains at least one nonarchimedean valuation and V has Witt index at least 1.

Bounded generation has connections to other properties of S-arithmetic groups. Let G be an algebraic group. Under certain assumptions $G_{\mathcal{O}_S}$ has bounded generation if and only if G satisfies the congruence subgroup property with respect to S [4].

Assume G is a group such that each of its finite index subgroups have finite abelianization. Then if G has bounded generation it is SS-rigid [5].

Not only does $SL_n(\mathbb{Z})$ have bounded generation, but it has the property with respect to unipotent elements, specifically the elementary matrices. This somewhat stronger property has been shown to be connected to the commensurated subgroup property. It has also been used in some cases to establish Kazhdan's Property (T) [1].

2 Completed Research

My dissertation research established bounded generation for two other families of S-arithmetic groups. The first family I considered extends Carter and Keller's result to the case of SL_n over an order of a quaternion algebra.

Theorem 1. *Let $\mathcal{D}$ be a quaternion algebra with standard basis $1, i, j, k$ over an algebraic number field K, $i^2 = \alpha$, $j^2 = \beta$. Let S be a finite set of valuations including all archimedean ones. Assume $\alpha, \beta \in \mathcal{O}_K$. Define*

$$\mathcal{O}_{\mathcal{D},S} = \{x_0 + x_1 i + x_2 j + x_3 k \mid |x_l|_v \leq 1 \textit{ for all } v \notin S, 0 \leq l \leq 3\}.$$

If S contains at least one nonarchimedean valuation, then $SL_n(\mathcal{O}_{\mathcal{D},S})$, the set of all matrices in $M_n(\mathcal{O}_{\mathcal{D},S})$ with reduced norm 1, has bounded generation for all $n \geq 2$.

This was proved by first showing that $SL_n(\mathcal{O}_{\mathcal{D},S})$ for $n \geq 2$ has bounded generation if $SL_2(\mathcal{O}_{\mathcal{D},S})$ does, since any element $X \in SL_n(\mathcal{O}_{\mathcal{D},S})$ can be expressed as a product of at most $2n^2 + n - 10$ elementary matrices and a block diagonal matrix consisting of an element of $SL_n(\mathcal{O}_{\mathcal{D},S})$ and an identity matrix. Bounded generation of $SL_2(\mathcal{O}_{\mathcal{D},S})$ was then proved by exhibiting an isomorphism $SL_2(\mathcal{D})$ to $\mathrm{Spin}(V)$ for a certain 6-dimensional vector space V.

I also considered unitary groups. Let L be a field with $[L : K] = 2$, and let τ denote the nontrivial element in the galois group. Let S_K be a set of valuations on K, and S_L be the set of all vaulations on L which restrict to elements of S_K. For a hermitian matrix $F \in M_n(\mathcal{O}_{S_L})$ we define $SU_{n,F}(\mathcal{O}_{S_L}) = \{X \in M_n(\mathcal{O}_{S_L}) \mid \tau(X^t)FX = F\}$

Theorem 2. *Let $a, b \in \mathcal{O}_{S_K}$, and let F be a diagonal matrix with entries $(a, -a, b, -b, \pm b, \ldots, \pm b)$, then $SU_{n,F}(\mathcal{O}_{S_L})$ has bounded generation.*

This was proved by reducing to the case of $SU_{4,F}$ and then showing that this is isomorphic to a Spin group. The restriction that $F = \mathrm{diag}(a, -a, b, -b)$ comes from the requirements of the isomorphism. However, while the reduction step for $SL_n(\mathcal{D})$ involved mostly basic linear algebra and matrix manipulations, the reduction step for $SU_{n,F}$ required a variety of interesting properties of arithmetic and algebraic groups, including strong approximation.

3 Future Work

Continuing in this direction I wish to establish that more general unitary groups have bounded generation. In particular, it can be shown that $U_{2,F}(\mathcal{O}_{\mathcal{D},S})$ has bounded generation for a hermitian matrix F and a quaternion algebra $\mathcal{D}$, which suggests that $U_{n,F}(\mathcal{O}_{\mathcal{D},S})$ may have bounded generation for larger n, since in the previous cases bounded generation was established by induction on n. It would also be interesting to try to establish whether $SU_{4,F}$ has bounded generation in the case where F has a nonsquare determinant.

While I would like to continue working on these problems, I am also interested in other areas. In particular, I attended the MSRI summer workshop on commutative algebra and have become fascinated by Boij-Soderberg theory. This theory describes the Betti diagrams of graded modules over the polynomial ring up to multiplication by a rational number. I am particularly interested in whether the Boij-Soderberg decomposition of certain modules can be related to geometric properties.

References

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