Math 126 Writing Up Problem 3: Taylor Series Applications
DUE WEDNESDAY, JUNE 1

1. (Evaluating Integrals and Limits)

   (a) Give $\int e^{x^2}\, dx$ as a Taylor series.

   (b) Use your answer from above to evaluate the integral $\int_0^1 e^{x^2}\, dx$ accurate to 5 digits after the decimal point (just keep using more terms until the 5 digits after the decimal stop changing).

   (c) Evaluate the limit $\lim_{x\to 0} \dfrac{x\sin(3x) - x^2}{\sin(5x^2) + 5x^2\sin(x)}$ by replacing $\sin(3x)$, $\sin(2x^2)$, and $\sin(x)$ by their Taylor series based at zero.

2. (Finding the value of infinite sums)

   (a) Give the values of the following sums by substituting into the appropriate Taylor series (assume they all converge):

   (i) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n$ (ii) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}$ (iii) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}$

   (iv) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!}$ (v) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ (vi) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$

   Aside: The last sum, (vi), can be used to approximate the value of π. It converges very slowly. That is, you have to go way out to get a lot of digits of accuracy, but there are adjustments you can make to get faster convergence. This has been a major method for approximating many digits of π in the past.

   (b) Do question 5 from page 1024 (Problems Plus) of your book.

   Aside: The sum $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, but the sums $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$, *etc* are all known to converge. Finding explicit values for these sums has been a source of great interest in the history of mathematics. In 1644, Pietro Mengoli posed the problem of finding the value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. In 1735, Leonhard Euler solved the problem and found that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. Euler is one of the greatest and most prolific mathematicians of all time. He published over 900 works spanning nearly all areas of math and science. He created many of the notations and conventions used today and for a good portion of his life he was blind. Our book discusses a solution to this problem on page 1024 (problems 5 and 6). We can't quite do problem 6 with the material we've covered, but you can read it if you are interested. Euler used a different method, but he still made use of Taylor series as part of his solution. Generalizing his solution, he found the explicit values for ALL even exponents. Explicit values for the odd exponents are still unknown and it is quite difficult to prove things about them.

3. (Simplifying *Messy* Functions) According the Einstein's theory of special relativity if an object at rest has mass m_0, then it's kinetic energy is given by

$$K = m_0 c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right).$$

According to the theory of classic Newtonian physics, the kinetic energy is given by

$$K = \frac{1}{2} m_0 v^2.$$

   (a) By differentiating with respect to v, find the first two nonzero terms of the Taylor series for $\frac{1}{\sqrt{1-v^2/c^2}}$ based at zero. Then show that if you only use these terms, that the two formulas are the same (for velocities much smaller than the speed of light, the higher order terms are negligible, so these formulas are about the same for small velocities). There are many other examples like this. Read problems 33-37 of Section 11.11 of your book for more examples.

4. (Solving Differential Equations) If I told you that $y = ax^2 + bx + c$ was a solution to $y' = 6x + 7$ you could differentiate y and compare coefficients to get $2ax + b = 6x + 7$. From that you could conclude that $2a = 6$, so $a = 3$ and $b = 7$. Thus, $y = 3x^2 + 7x + c$ (but you already knew this). This same idea can be used to find solution to more complicated differential equations using Taylor series.

   Here is an example: We know, from Math 125, how to solve the initial value problem $\frac{dy}{dx} = y$ with $y(0) = 1$. You might remember the solution. Now let's solve it another way.

   Let $y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$ be the unknown Taylor series for $y(x)$. Since $y(0) = 1$, we know that $a_0 = 1$.

   Differentiating we get, $y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Putting these in the differential equation $\frac{dy}{dx} = y$, we get $a_1 + 2a_2 x + 3a_3 x^2 + \cdots = a_0 + a_1 x + a_2 x^2 + \cdots$. Now we equate coefficients to get

$$\begin{aligned} a_1 &= a_0 = 1, && \text{(by the initial condition).} \\ 2a_2 &= a_1 = 1, && \text{so } a_2 = \tfrac{1}{2}. \\ 3a_3 &= a_2 = \tfrac{1}{2}, && \text{so } a_3 = \tfrac{1}{6} = \tfrac{1}{3!}. \\ 4a_4 &= a_3 = \tfrac{1}{3!}, && \text{so } a_4 = \tfrac{1}{4!}. \end{aligned}$$

   And, in general, we are seeing that $a_n = \frac{1}{n!}$. Thus, the answer is $y = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, which we know from class is e^x. This may seem like a roundabout method, but it is very useful because it is a general method that can be used even when the differential equation is not separable.

   (a) Try the method, on the following differential equation $y' = x + y$ with $y(0) = 0$. This is not separable, so you currently have no method from Math 125 to solve this. Let's try to solve it with Taylor series. Give the first 5 nonzero terms of the Taylor series for the answer using the method described in the previous example. You should recognize your answer. Rewrite your final answer in terms of known functions.