Wave and Heat Equations and Fourier Series
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The notes below are definitions and examples, which we will eventually turn into a project of some sort.

1 Wave Equation

Recall the 1D wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

where c is the wave speed and $u(x,t)$ denotes the distance that the string at location x, for $0 \leq x \leq L$, is displaced from equilibrium at time t. A string that is initially plucked (with no initial velocity, i.e., not "struck") that is held taught at both ends is modeled by the initial and boundary conditions

$$u(0,t) = u(L,t) = 0, \quad \frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(L,t) = 0,$$
$$u(x,0) = \phi(x), \quad \frac{\partial u}{\partial t}(x,0) = \psi(x) = 0.$$

The solutions, consisting of the kth modes of vibrations, turn out to be:

$$u(x,t) = \sum_{k=0}^{\infty} A_k \cos\left(\frac{k\pi ct}{L}\right)\sin\left(\frac{k\pi x}{L}\right),$$

so that at $t=0$, the initial displacement becomes

$$\phi(x) = u(x,0) = \sum_{k=0}^{\infty} A_k \sin\left(\frac{k\pi x}{L}\right).$$

The coefficients A_k come from the Fourier sine series for $\phi(x)$, which implies

$$A_k = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{\pi k x}{L}\right)dx.$$

See the next page for the heat equation which uses similar solution methods (Fourier Series)

2 Heat Equation

Rod: $\dfrac{\partial u}{\partial t} = \alpha \dfrac{\partial^2 u}{\partial x^2}$ Region: $\dfrac{\partial u}{\partial t} = \alpha \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$ Solid: $\dfrac{\partial u}{\partial t} = \alpha \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} \right)$

where

- α = thermal diffusivity
- Laplacian: $\Delta = \nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

Solutions to $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$ are called Harmonic Functions.

1. $u_t = \alpha u_{xx}$, where $0 \leq x \leq L$ is the location the rod and t is time.
2. Initial conditions
   - $u(x, 0) = f(x)$, initial temp at each point
   - $u(0, t) = 0 = u(L, t)$, temp at ends kept at zero.
3. One Solution (given in Wikipedia article)
   - $u(x, t) = \displaystyle\sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2\alpha t}{L^2}}$
   - where $D_n = \dfrac{2}{L} \displaystyle\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$