Torricelli's Law and How Long to Drain a Bathtub

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Concepts and usage: Integration and basic intro to differential equations. Best used in the last two weeks of Math 125 at UW.

Introduction: Assume you have a large container full of water. The container has a small hole in the side and water is streaming out. In 1643, an Italian physicist named Evangelista Torricelli wrote in an appendix to one of his major works, "Opera Geometrica," about this very situation. The translation of his main assumption states...

> "Let us suppose that the water which exits violently from an orifice has... the same impetus as a heavy body... which falls from the liquid's surface until the orifice."."

The word "orifice" is out of fashion, but the model is very accurate. For a tank with constant cross-sectional area, this assumption leads to the following differential equation for the height of the water above the hole:

$$\frac{dh}{dt} = \frac{A_H\sqrt{2g}}{A}\sqrt{h}, \text{ where}$$

- A is the cross-sectional area of the tank, A_H is the area of the hole, and g is the constant for acceleration due to gravity.

(See the next page for a derivation)

Playing Around with the Model

1. *Emptying Your Bathtub* Your bathtub is full of water and you pull the plug for the drain. Here are some measurements: The drain at the bottom of your bathtub is circular with a radius of 0.02 meters. The cross-section area of your bathtub is a constant 0.6 square meters. The bathtub is initially filled to a height of 0.4 meters. Use $g = 9.8$ m/s^2.

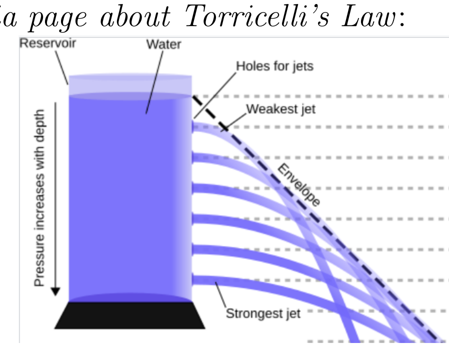
   (a) Solve the differential equation for $h(t)$.

   (b) How long will it take for the bathtub to empty? (Note that t is in seconds, please convert it to minutes).

2. *How far will the stream go* Assume you are a particular hole in the side of a cylindrical tank such as in the picture below. Assume water comes out of the hole with an initial horizontal velocity of $\sqrt{2gh}$, where h is the height of the water above the hole, as predicted by Torricell's Law (see next page for details).

   (a) If the hole is a distance of h_t from the top of the water and a distance of h_b from the ground. Then how far horizontally will the water have reached when it hits the ground?

Here is a visual from the Wikipedia page about Torricelli's Law:



Derivation of the Model:

Torricelli's assumption means that the following two speeds are the same:

- The velocity of an object when it hits the ground when dropped from height h.
- The velocity of the water coming out of the hole of the tank if the water level is at a height h above the hole.

1. *Velocity of the Water*: If an object is *dropped* from height h meters and falls at a constant rate of g meters/second, then the velocity when it hits the ground is $v = \sqrt{2gh}$. Verify this fact by starting with $a(t) = -g$, $v(t) = -gt + c$, $y(t) = -\frac{1}{2}gt^2 + ct + d$, then putting in the initial conditions, solving for when $y(t) = 0$, then getting the velocity at this time.

2. *Rate of Change of Volume*: The amount of water that exits through the hole each second, which is $\frac{dV}{dt}$, is given by $A_H v$ where A_H is the cross-sectional area of the hole and v is the velocity you just found. So we have
$$\frac{dV}{dt} = A_H v = A_H\sqrt{2gh}$$

3. *A differential equation for h*: Assume we have a tank with constant cross-sectional area A, which means $V = Ah$ and $\frac{dV}{dt} = A\frac{dh}{dt}$. Putting this fact together with the previous fact gives.
$$A\frac{dh}{dt} = \frac{dV}{dt} = A_H\sqrt{2gh} \Longrightarrow A\frac{dh}{dt} = A_H\sqrt{2gh}.$$

   For simplicity, let's write this differential equation in the form
$$\frac{dh}{dt} = k\sqrt{h}, \text{ where } k = \frac{A_H}{A}\sqrt{2g}.$$