Surface Area and Soap Bubbles

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Concepts and usage: Introduces another use of integrals not covered in Math 125 and mentions concepts from upper-level math courses on minimal surfaces.

Introduction: In Math 125, for a function $y = f(x)$ from $x = a$ to $x = b$ we derived the integral for the length of the curve as $\displaystyle\int_a^b \sqrt{1 + (f'(x))^2}\, dx$. If you take that same portion of the curve and rotate it around the y-axis, then a similar derivation gives the surface area of the resulting surface as $\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$ (which can also be computed in terms of y). In this project we will compute some surface areas, for more info see 8.2 of Stewart's calculus text (or search calculus surface area online).

1. Classic Examples:

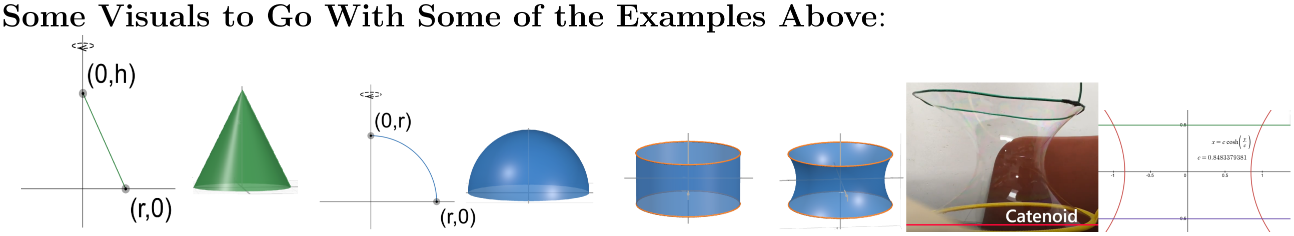
   (a) The Cone: Consider the line segment from $(0, h)$ and $(r, 0)$ where r and h are positive constants. Use the integral definition to find the surface area for the surface obtained by rotating this line about the y-axis.

   (b) The Sphere: Consider the part of the circle $x^2+y^2 = r^2$ in the first quadrant. Use the integral definition to find the surface area for the surface obtained by rotating this curve about the y-axis. Then use that to give the surface area of the sphere.

2. Minimal Surface Area:

   - Set-Up: Assume you have two circular rings of radius 1 that are parallel and one unit apart. Let's consider a couple surfaces that go through both these rings. Consider a circular cylinder of radius 1 that goes through both these rings. The surface area of that cylinder is given by $2\pi rh$ where $r = 1$ and $h = 1$, so we have a surface area of 2π square units. An interesting fact is that this is not the smallest connected area you can get from a surface through these rings!
   - Info: An interesting fact is that the circular cylinder is not the smallest connected area you can get from a surface through these rings! In fact, if you place both rings in a soap mixture and slowly pull them apart, then the soap bubbles forms a "minimum surface area" and it is not a cylinder, see below. The shape you get is called a Catenoid. And there is a limit to how far apart they can be before the bubble pops (see an examples below). The catenoid is what you get when you draw $y = c \cdot \cosh(y/c)$, for some constant c, then rotate about the y-axis, where $cosh(y) = \frac{1}{2}\left(e^y + e^{-y}\right)$.
   - Problem: Numerically, we found that the $c = 0.8483379381$ works to go through two rings that are one unit apart and of radius 1 (see below). Set up the integral for the surface area. Then using an online integrator, find the numerical value for the surface area of this catenoid and verify that it is smaller than 2π.

Some Visuals to Go With Some of the Examples Above:



A more general formula for surface area for any surface $z = f(x, y)$ over a region R is given by $\iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dx$ but that requires a small bit of calculus III.