Studying Salmon Population at Ballard Locks

A Brief History of the Ballard Salmon Ladder:

In 1891, the United States Army Corps of Engineers began the process of designing a canal system in Lake Washington, its construction being successfully completed in 1916. With the introduction of the new canal, the salmon population could no longer migrate to spawn in the summertime. In efforts to maintain the crucial salmon population of Washington, cement salmon ladders were designed and constructed, offering an artificial route for salmon to migrate and reproduce each year.

Concepts & Usage:

The logistic growth model is used to show the relationship between population growth and time. It consists of a constant "k" value, known as the carrying capacity, which is the maximum population size that can exist with limited resources.

$$Logistic\ growth\ model\colon \frac{dP}{dt} = kP(1 - \frac{P}{k})$$

Using this model, we will estimate and analyze salmon populations within Ballard Locks' salmon ladders.

Mathematical Applications:

1.) Everyday, researchers count the number of sockeye salmon living at the Ballard Locks, to record the population increase during the change of the seasons. They assume that the rate the salmon population increases is proportional to the size of the population modeled by the equation dp/dt = kp, where k is a constant value known as the carrying capacity and p is the amount of salmon. On the initial day of research (day 0) , there was one salmon at the ladder. On the third day of research, the scientists see that there are 7 new fish at the locks. Solve the differential equation for p(t), which represents the salmon population.

2.) The von Bertalanffy growth equation is a model describing the growth of fish in a population, where L_x is the maximum length of a sockeye salmon recorded in the Ballard Locks, L represents the length of the fish in inches on a certain day (t), and k is a constant that determines how the fish grows.

$$\frac{dL}{dt} = k(L_x - L) \text{ \& } L(0) = 12$$

   a.) Solve the differential equation knowing the maximum length of the salmon is 30 inches
   b.) Solve for the k-value assuming that L(4) = 14
   c.) Solve for the length of the fish at day 10

Solutions:

1.) $dt \cdot \frac{dP}{dt} = kP \cdot dt$ $P(0)=1$
$P(3)=7$

$dp = kp\,dt$

$\int \frac{1}{p}\,dp = \int k\,dt$

$e^{\ln|p|} = e^{kt+c}$

$P(t) = e^{kt} \cdot e^{c}$

$P(t) = Ae^{kt}$

$A = e^{c} = 1$ → $P(t) = e^{kt}$

$P(0) = 1 = e^{c} \cdot 1$

$e^{c} = 1$

$P(3) = 7$

$\ln 7 = \ln e^{3k}$

$\ln 7 = 3k$

$k = \frac{\ln 7}{3}$

→ $P(t) = e^{\left(\frac{\ln 7}{3}\right)t}$

2.) $\frac{dL}{dt} = k(30-L)$

a.) $\int \frac{dL}{30-L} = \int k\,dt$

$-\ln|30-L| = kt + c$

$e^{\ln|30-L|} = e^{-kt+c}$

$|30-L| = e^{-kt+c_1}$

$|30-L| = c_2 e^{-kt}$

$30-L = \pm c_2 e^{-kt}$

$30-L = c_3 e^{-kt}$

$L(t) = 30 - c_3 e^{-kt}$ → $L(0) = 12$

$12 = 30 - c_3 e^{0}$

$12 = 30 - c_3$

$c_3 = 18$

$L(t) = 30 - 18e^{-kt}$

b.) $L(4) = 14$

$14 = 30 - 18e^{-4k}$

$-16 = -18e^{-4k}$

$\frac{8}{9} = e^{-4k}$

$\ln\left(\frac{8}{9}\right) = -4k$

$4k = -\ln\left(\frac{8}{9}\right)$

$\frac{4k}{4} = \frac{\ln\left(\frac{9}{8}\right)}{4}$ → $k = \frac{\ln\left(\frac{9}{8}\right)}{4}$

$L(t) = 30 - 18e^{\left(\frac{-\ln\left(\frac{9}{8}\right)}{4}\right)t}$

c.) $L(10) = 30 - 18e^{\left(\frac{-\ln\left(\frac{9}{8}\right)}{4}\right)10}$

$= 16.591$ inches