Population Models
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Concepts and usage: Differential Equations. Could be asked in the last two weeks of Math 125.

Introduction: Let $P(t)$ be the population of some group at time t. The study of population growth is very important in the sciences. We use these models to estimate future spread of a disease or the growth of bacteria or the number of people in a city. These estimates often lead to important decisions about preparing for the future. In this project we will explore a few starting ideas from population modeling in hopes to encourage you to play around with these ideas.

Unrestricted Growth

Let $P(t)$ be the population of a city after t years. Note that $\frac{dP}{dt}$ represents the rate of change of population with respect to time in people per year. **For the models below, assume you are studying the population of a city that currently has 5,000 people.**

1. *Constant growth*: Assume the city is growing at a constant 100 people per year, then the differential equation for the change is $\frac{dP}{dt} = 100$ with $P(0) = 5000$. Solve the differential equation for $P(t)$.

2. *Natural growth*: Assume the city is growing at a rate proportional to the population size, then the differential equation for the change is $\frac{dP}{dt} = kP$ with $P(0) = 5000$. Also assume that in one year the population is $P(1) = 5100$ people. Solve the differential equation for $P(t)$.

3. *Natural growth with immigration*: Assume the city is growing at a rate proportional to the population size and that 20 people immigrate into the city each year, then the differential equation for the change is $\frac{dP}{dt} = kP + 20$ with $P(0) = 5000$. Also assume in one year that the population is $P(1) = 5100$. Solve the differential equation for $P(t)$.

4. For each model above, how long will it take for the poulation to get to 8,000 people?

Restricted Growth

Assume again that a town has 5,000 people, but now assume the population size isn't changing. At the start of the year, 5 people in the town have a very contagious virus that causes a mild cold. After 5 days, 40 people in the town have the virus. Let $y(t)$ be the number of sick people after t days.
The growth is like natural growth original, but the number of sick people can't exceed 5,000 people because there are only 5,000 people in the town. The most commonly used and studied model for this situation is the logistics model given by:

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{5000}\right), y(0) = 5, y(5) = 40$$

1. Find the two values of y at which $dydt$ is equal to zero. What does this tell you about the slope of the model when y gets close to the size of the city?

2. Solve the differential equation, then graph the resulting solution. The solution to this model is called a sigmoid function, see our other project on these types of functions.

3. At what time is the virus spreading fastest?

Going further?

1. *Other models?* Another very commonly used model that is supposedly good for modeling the spread of a disease or the growth of a tumor is the Gompertz Model. The model used on the same problem from the previous page could be given by $\frac{dy}{dt} = ry \ln\left(\frac{5000}{y}\right)$. Solve this model and compare results.

2. *Two populations at the same time?* Another very common type of model in biology is a 'coupled' system where two populations (or chemicals) interact with each other and effect each other. For example, if $x(t)$ is the number of rabbits in a forest after t months and $y(t)$ is the number of wolves in the same forest, then one might expect that the the two populations will effect each other in various ways. There are many examples of this, but a famous model is the Lotka-Volterra Model which aims to study predator-prey systems like this and is given by

   - $\frac{dx}{dt} = ax - bxy$
   - $\frac{dy}{dt} = cxy - dy$
   - where a, b, c and d are positive constants.

   The xy terms give a measure of all possible rabbit and wolve interactions and if that is large, then we might expert more rabbits are getting eaten (which means more wolves are reproducing). If this interests you let me know, we can help you ask questions about these models and explore them. A common questions where the system will reach some sort of repeating equilibrium or if the rabbits or wolves will die out.