Title:

Calculus about moon’s orbit

Concept and usage:

This project uses calculus and Kepler’s laws to model the Moon’s orbit, transitioning from a simplified circular model to a more accurate elliptical one.

The students could practice integral and derivative skills when calculating orbital properties like period and area swept by the moon.

Introduction:

By combining Kepler’s Laws of Planetary Motion with calculus, we are able to predict the moon's position, velocity and acceleration in the key.

Kepler’s First Law: Modeling the Moon’s elliptical orbit.

Kepler’s Second Law: Using calculus to prove equal areas are swept in equal times.

Kepler’s Third Law: Relating the orbital period to distance.

Questions:

First, we will assume a circular orbit. In reality, the moon’s orbit is slightly elliptical with an eccentricity e = 0.055. But it is close to circular.

1) **How do we get the period of the moon’s orbit about the Earth?**

Some values we need to know:

Mass of the moon = m1 = 7.36 * 10^22 kg

Mass of Earth = m2 = 5.97 * 10^24 kg

Gravitational constant = G = 6.674 * 10^(-11) (N*m^2)/kg^2

Mean distance of earth to moon = 3.84 * 10^8 m

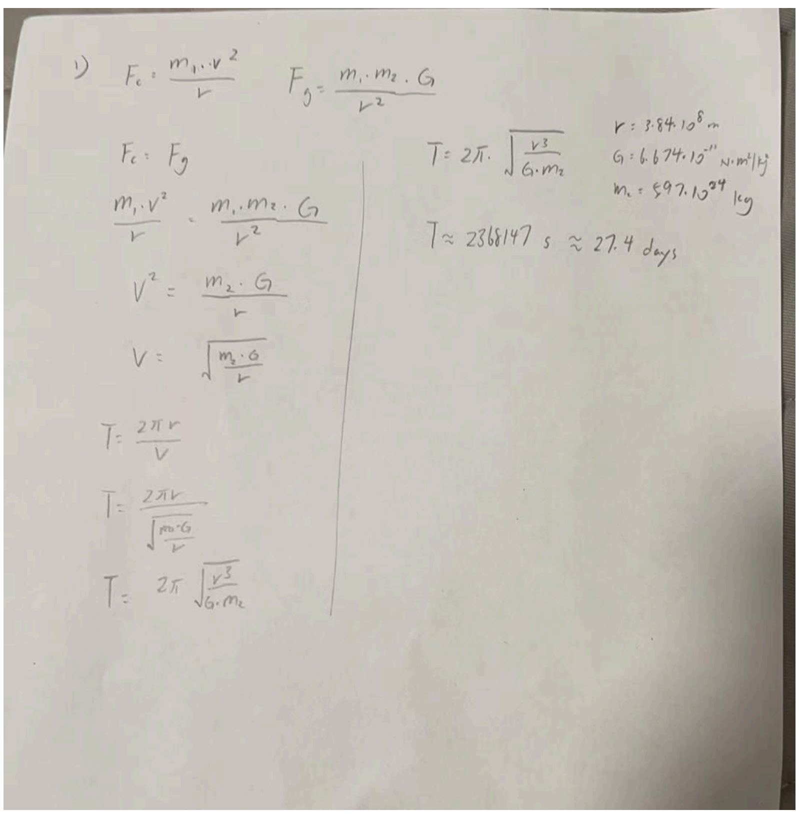
Moon's orbital eccentricity = 0.055

First equation: T = (2 * pi * r) / v

Second equation: Fc = (m1 * v^2) / r

Third equation: Fg = (G * m1 * m2) / r^2

And the gravitational force is acting as the centripetal force.



1) $F_c = \frac{m_1 \cdot v^2}{r}$ $F_g = \frac{m_1 \cdot m_2 \cdot G}{r^2}$

$F_c = F_g$

$\frac{m_1 \cdot v^2}{r} = \frac{m_1 \cdot m_2 \cdot G}{r^2}$

$v^2 = \frac{m_2 \cdot G}{r}$

$v = \sqrt{\frac{m_2 \cdot G}{r}}$

$T = \frac{2\pi r}{v}$

$T = \frac{2\pi r}{\sqrt{\frac{m_2 \cdot G}{r}}}$

$T = 2\pi \sqrt{\frac{r^3}{G \cdot m_2}}$

$T = 2\pi \cdot \sqrt{\frac{r^3}{G \cdot m_2}}$

$r = 3.84 \cdot 10^8$ m

$G = 6.674 \cdot 10^{-11}$ $N \cdot m^2/kg^2$

$m_2 = 5.97 \cdot 10^{24}$ kg

$T \approx 2368147$ s ≈ 27.4 days

2) **If the moon orbits the Earth circularly, find its velocity vector at t = 5 days.**

2)

$x(t) = R\cos(wt)$ $\quad$ $y(t) = R\sin(wt)$

$w = \frac{v}{r} = \frac{2\pi}{T} = \frac{2\pi}{27.3} \approx 0.23 \text{ rad/day}$

Position $\quad$ $x(t) = 384400 \cos(0.23t)$

$y(t) = 384400 \sin(0.23t)$

velocity $\quad$ $V_x(t) = \frac{dx}{dt} = -Rw\sin(wt)$

$V_y(t) = \frac{dy}{dt} = Rw\cos(wt)$

$t = 5$

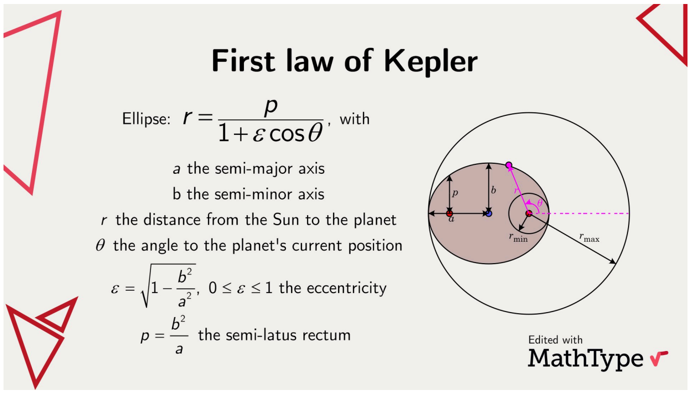
$V_x = -384400 \cdot 0.23 \cdot \sin(0.23 \cdot 5) \approx -80800$ km/day

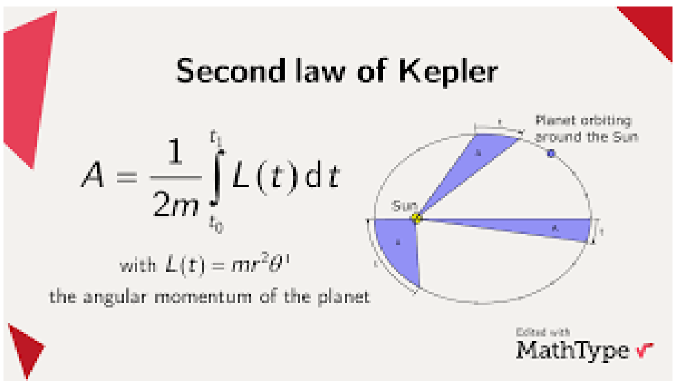
$V_y = 384400 \cdot 0.23 \cdot \cos(0.23 \cdot 5) \approx 36100$ km/day

Acceleration $\quad$ $a_x(t) = \frac{dv_x}{dt} = -Rw^2\cos(wt)$

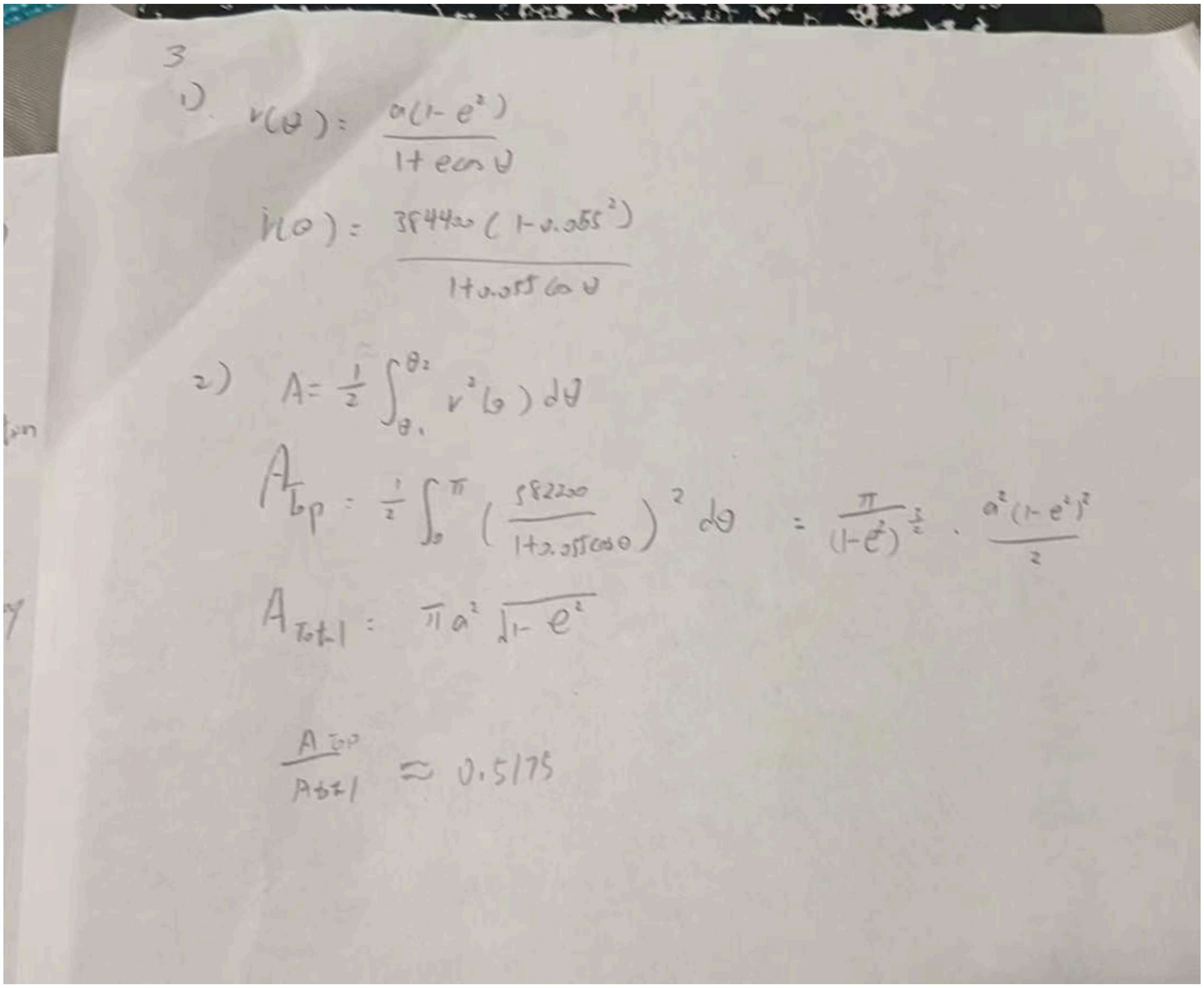
$a_y(t) = \frac{dv_y}{dt} = -Rw^2\sin(wt)$

3) **Apply the Kepler's laws for more question**





For example: how much time the Moon spends in the top half of its orbit



3

1) $r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta}$

$r(\theta) = \frac{384400(1-0.055^2)}{1+0.055\cos\theta}$

2) $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2(\theta)\,d\theta$

$A_{bp} = \frac{1}{2}\int_0^{\pi}\left(\frac{582200}{1+0.055\cos\theta}\right)^2 d\theta = \frac{\pi}{(1-e^2)^{\frac{3}{2}}}\cdot\frac{a^2(1-e^2)^2}{2}$

$A_{Total} = \pi a^2\sqrt{1-e^2}$

$\frac{A_{top}}{A_{total}} \approx 0.5175$

Since Kepler's second law taught us that the moon sweeps equal areas in equal times.

The Ttop/Ttotal is also 0.5175.

Visual:

https://www.mooncalc.org/#/49.495,11.073,3/2025.08.17/23:41/1/3