# Estimating Moments of Functions of Random Variables with Taylor Expansions

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## 1 Concept

One topic you gain access to with the tools of Calculus is the topic of Probability. Probability is a branch of mathematics that is concerned with events and measuring their likelihood of occurring. These events can be discrete in nature, such as the flip of a coin; or continuous, such as selecting a random real number between 0 and 1. Often times, mathematicians are concerned with measuring a random variable, which is a function that maps "outcomes" to some measurable space (usually the real number line). What makes calculus useful in this context is tools such as summation and integration allow us to make better sense of an infinite number of outcomes, as is the case for continuous random variables.

One measure we are particularly concerned with are the moments of a random variable. Most people are familiar with moments from physics such as torque, but in probability they have a different meaning. Moments in probability refer to the shape of a function's graph, which in this case is the graph of its distribution (a function that gives the probabilities of occurrence of outcomes a random variable can take). Moments are helpful as they can help us find out important measures of a random variable's distribution, such as its mean (average) and its variance.

For a continuous random variable $X$, its $n$th raw moment is defined as follows:

$$\mu_n' = \mathbb{E}[X^n] = \int_\Omega x^n f(x)\mathrm{d}x$$

Most commonly though, moments are found using a moment generating function (MGF):

$$M_X(t) = \mathbb{E}[e^{tX}]$$

$n$th moments can then be derived from the MGF by taking the $n$th derivative and evaluating at 0:

$$\mu_n' = M_X^{(n)}(0)$$

What we are interested in however, is an approximation. At times, random variables can have quite unwieldy distributions, which makes evaluating an integral challenging. In this case, we can use a taylor series to approximate a moment, rather than try and solve an integral.

## 2 Examples of Taylor Expansion for Mean and Variance

We can use a Taylor Series to approximate the mean of a function of a random variable, here is an example:

$$\mathbb{E}[f(X)] = \mathbb{E}[f(\mu_X + (X - \mu_X))]$$

$$\mathbb{E}[f(\mu_X + (X - \mu_X))] \approx \mathbb{E}\left[f(\mu_X) + f'(\mu_X)(X - \mu_X) + \frac{1}{2}f''(\mu_X)(X - \mu_X)^2\right]$$

In this case, we only used three terms, but if we continued this series onward, we could get a more and more accurate approximation of the mean.

$$\mathbb{E}[f(X)] = \sum_{n=1}^{\infty} \frac{f^n(\mu_X)(X - \mu_X)^n}{n!}$$

Now observe for variance:

$$\text{Var}(f(X)) = \mathbb{E}[f(X)^2] - \mathbb{E}[f(X)]^2$$

We expand around $\mu_X$, keeping just the first two terms

$$f(X) \approx f(\mu_X) + f'(\mu_X)(X - \mu_X) + \frac{1}{2}f''(\mu_X)(X - \mu_X)^2$$

$$\text{Var}(f(X)) = (f'(\mu_X))^2 \text{Var}(X)$$

Variance is a bit messier, but to write it in series notation, we have:

$$\text{Var}(f(X)) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{f^{(i)}(\mu_X)f^{(j)}(\mu_X)}{i!j!}\mu_{i+j} - \left(\sum_{k=0}^{\infty} \frac{f^{(k)}(\mu_X)}{k!}\mu_k\right)$$

Where $\mu_a$ is the $a$th central moment (about the mean)

## 3 Example Problem

Suppose $X$ follows a standard normal distribution, approximate the mean of $f(X)$, where $f(x) = x^2$

We can use our first formula to do this:

$$\mathbb{E}[X^2] = \sum_{n=1}^{\infty} \frac{f^n(\mu_X)(X - \mu_X)^n}{n!}$$

If we write out the first three terms, we get:

$$\mathbb{E}[X^2] \approx \mu^2 + 2\mu(\mu - \mu) + \sigma^2 = \mu^2 + \sigma^2 = 1$$

So we have that the expectation of $X^2$ is one. This actually matches the exact answer if we were to just solve by integrating, since our function is a polynomial.