Melting Snowballs and Differential Equations
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Concepts and usage: Integration and basic intro to differential equations. Best used in the last two weeks of Math 125 at UW.

Introduction: A large spherical snowball is placed on a rack in my office and slowly melts over nine hours. A good model for melting snow is given by: **The snowball melts at a rate proportional to its surface area**. This assumptions leads to the differential equations for volume and radius below (for more details on where this comes from see the next page)...

$$\frac{dV}{dt} = -aV^{2/3} \quad \text{and} \quad \frac{dr}{dt} = -k \quad \text{, where} \quad a = 3^{2/3}\,(4\pi)^{1/3}\,k.$$

Playing Around with the Model

1. Give the general solution, $V = V(t)$, for $\frac{dV}{dt} = -aV^{2/3}$ by separating and solving.
2. Give the general solution, $r = r(t)$, for $\frac{dr}{dt} = -k$.
3. If snowball with an initial radius of 2.5 inches completely melted (*i.e.* the radius was zero) in 9.25 hours. Find the formulas for $r(t)$ and $V(t)$ for this example.
4. If the k values you just found it accurate, what does the initial volume of a spherical snowball need to be so that it will take exactly one day for it all to melt in my office?
5. *A note on theory from Math 207*: We say the differential equation for the volume is nonlinear because of the $V^{2/3}$ in the equation. Nonlinear equations can sometimes have unusual behaviors at 'equilibrium' values. For example, even for a given value of k, there is more than one solution to with $V(9.25) = 0$, can you think of another?

Visuals of a snowball melting in 9.25 hours and a QR code link to a video series on this project:

Details on the set-up: **The snowball melts at a rate proportional to its surface area**. This means that the rate at which the volume changes $\frac{dV}{dt}$ is some multiple (*think percentage*) of the exposed surface area.

- The surface area and volume of a sphere are given by $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$ where r is the radius. In this problem all three of these quantities are changing as a function of time t.
- A direct translation of our assumption is that for some constant k, we have $\frac{dV}{dt} = -kS$, which is the same as $\frac{dV}{dt} = -k4\pi r^2$.
- You try: Solve for r in terms of V and verify that we get $\frac{dV}{dt} = -k4\pi \left(\frac{3}{4pi}V\right)^{2/3}$, which is the same as $\frac{dV}{dt} = -aV^{2/3}$, where $a = k \cdot 3^{2/3} (4\pi)^{1/3}$.
- A much easier way to do this problem is to remember your related rates and note that $V = \frac{4}{3}\pi r^3$ implies $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. Which means that $\frac{dV}{dt} = -k \cdot 4\pi r^2$ and implies that $\frac{dr}{dt} = -k$.
- Thus this assumption leads to a situation where radius is changing at a constant rate!