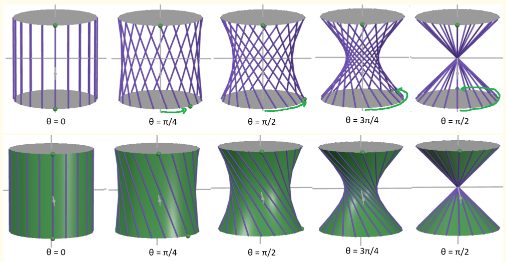
Work in progress
by Layan Arrabi

Concepts and usage: Volumes of Revolution and Surface Area. Best used in the third week of Math 125 at UW.



Introduction: The surface obtained by rotating the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1$ about the z-axis is called a hyperboloid of one sheet. In this project, we will explore how the volume inside a hyperboloid compares to the surface area of the hyperboloid while showing how hyperboloids can be formed from an infinite number of lines. The collection of all points (x, y, z) on this hyperboloid are given by $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$, where the z-axis is the axis of rotation.

Ruling Lines and Fun Facts: A *ruling line* is a line that is completely contained on a surface in 3D. A hyperboloid is unusual because it is a curved surface on which every point on the surface is on a ruling line. Another fun fact is that if you take two circular parallel rings, attach elastic string vertically between them, and slowly rotate the bottom, then all the lines formed by the strings will be on the same hyperboloid. Visuals at various angles of rotation, θ, are shown above.

1. **Volume of revolution**: Consider the region bounded by $z = H$, $z = -H$, where $H = h/2$ (h is the height), and $\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1$ such that $x \geq 0$. The volume of the solid obtained by rotating this region about the z-axis is

$$v = \pi \int_{-H}^{H} x(z)^2 dz = \pi a^2 (2H + \frac{2H^3}{3b^2}) = \pi a^2 h (1 + \frac{h^2}{12b^2})$$

2. **Volume in terms of θ**: Given that R is the radius of the bottom and top circle.

$$v = \frac{\pi R^2 h}{3}(2 + cos(\theta))$$

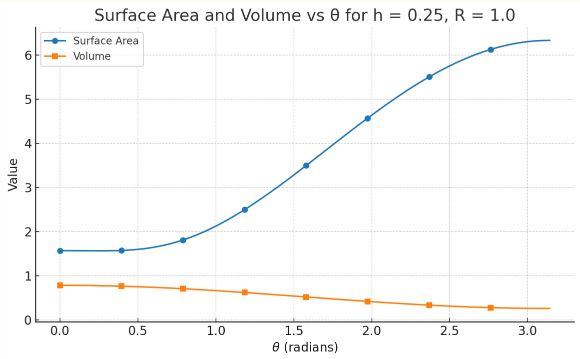
3. **Surface Area of Hyperboloid**: We will get the surface area of the one sheet hyperboloid by rotating $r(z) = a\sqrt{1 + \frac{z^2}{b^2}}$ from $z = -H$ to $z = H$. Thus, the surface area is

$$S = 2\pi \int_{-H}^{H} r(z)\sqrt{1 + (r'(z))^2} dz = 2\pi a [H\sqrt{1 + \frac{(a^2 + b^2)H^2}{b^4}} + \frac{b^2}{\sqrt{a^2 + b^2}} \sinh^{-1}(\frac{H\sqrt{a^2 + b^2}}{b^2})]$$

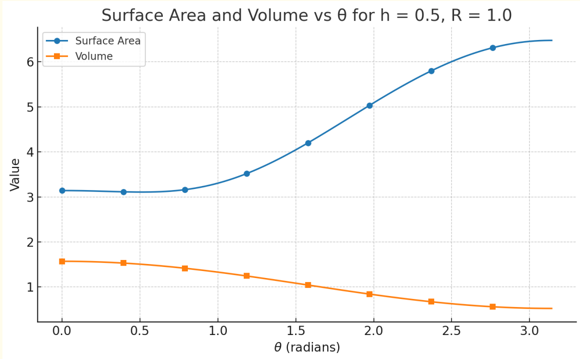
4. **Surface Area in terms of θ**: Given that R is the radius.

$$S(\theta) = 2\pi R H \sqrt{1 + \frac{R^2}{H^2}\sin^4\left(\frac{\theta}{2}\right)} + 2\pi R \frac{H^2 \cos^2\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)\sqrt{R^2 \sin^2\left(\frac{\theta}{2}\right) + H^2}} \sinh^{-1}\left(\frac{\sin\left(\frac{\theta}{2}\right)}{H\cos\left(\frac{\theta}{2}\right)}\sqrt{R^2\sin^2\left(\frac{\theta}{2}\right) + H^2}\right).$$

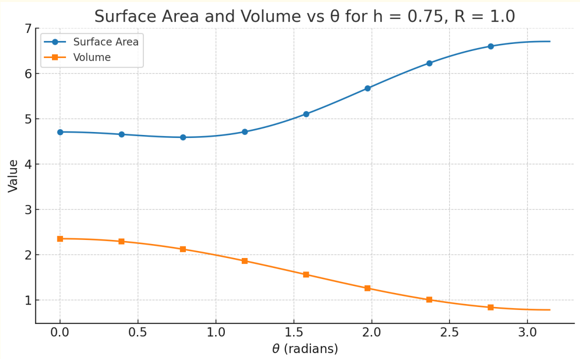
Main Question: How differently does the surface area of the hyperboloid change compared to the volume as we rotate by θ? We will compare them at different heights and see if there are any differences.



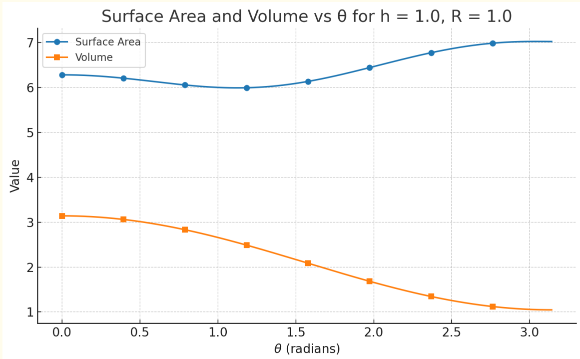
(a) $h = \frac{1}{4}$



(b) $h = \frac{1}{2}$



(c) $h = \frac{3}{4}$



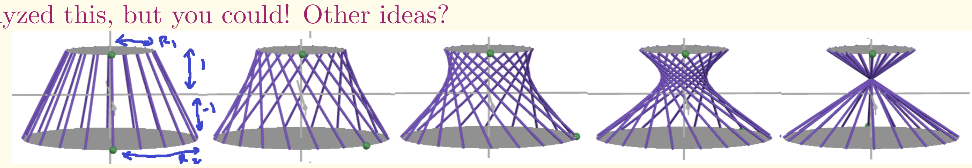
(d) $h = 1$

Figure 1: Comparison of Surface Area and Volume vs. θ for $R = 1$ at different heights h.

Reflection: The output of the graphs is very interesting. The volume of the hyperboloid appears to decrease as we twist. It also seems to me that the volume changes very predictably as the height increases, unlike the surface area. The surface area increases the more we twist the hyperboloid. The surface area curves also look very different from graph to graph. Another interesting thing happening is that the larger the height the larger the difference is between the surface area and the volume. There is also less change in surface area the larger the height is.

Other Questions

1. For $0 \leq \theta \leq \pi$, at what angle is the volume largest? At what angle is the volume smallest? At what angle is the volume changing the fastest?
2. What if you change the radius of the bottom plate to something bigger than 1? We have not analyzed this, but you could! Other ideas?



3. Do surface area and volume ever intersect? If so, at what value of θ for a fized h?
4. Given a fixed volume, at what θ is the surface area minimized? Conversely, given a fixed surface area, when is the volume maximized?
5. Hyperboloids are used in cooling towers and other structures. Why?