Volume of Hyperboloids and Ruling Lines

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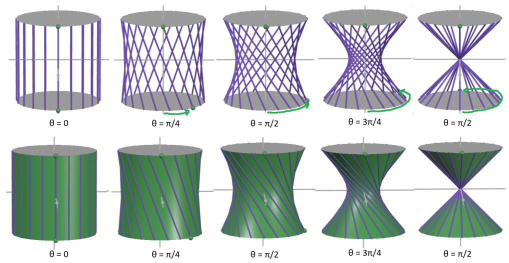
Concepts and usage: Volumes of Revolution. Best used in the third week of Math 125 at UW.



Introduction: The surface obtained by rotating the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1$ about the z-axis is called a hyperboloid of one sheet. In this project we will explore the volume inside a hyperboloid while showing how hyperboloids can be formed from an infinite number of lines. The collection of all points (x, y, z) on this hyperboloid are given by $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$, where the z-axis is the axis of rotation.

Ruling Lines and Fun Facts: A *ruling line* is a line that is completely contained on a surface in 3D. A hyperboloid is unusual because it is a curved surface on which every point on the surface is on a ruling line. Another fun fact is that if you take two circular parallel rings, attach elastic string vertically between them, and slowly rotate the bottom, then all the lines formed by the strings will be on the same hyperboloid. Visuals at various angles of rotation, θ, are shown above.

1. **Volume of revolution**: Consider the region bounded by $z = 1$, $z = -1$, and $\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1$ such that $x \geq 0$. Find the volume of the solid obtained by rotating this region about the z-axis.

2. **Volume in terms of θ**

   (a) First, since $(1, 0, 1)$ is a point on $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$ we can substitute this to our equation to get $\frac{1}{a^2} - \frac{1}{b^2} = 1$. Use this to find an equation for b^2 in terms of a^2.

   (b) Second, we will find a formula for a in terms of θ.

   - Consider the string that has one point fixed at $(1, 0, 1)$. After rotating the bottom ring counterclockwise by an angle θ, the other end of the string will be at the point $(\cos(\theta), \sin(\theta), -1)$. Find the midpoint of $(1, 0, 1)$ and $(\cos(\theta), \sin(\theta), -1)$.
   - The value of a in $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$ represents the radius of the circular intersection of the hyperboloid and the xy-plane. The midpoint you just found is on this circle. Find the distance from $(0, 0, 0)$ to the midpoint, which gives a formula for a in terms of θ.

   (c) Give the volume in terms of θ by replaying a and b in your volume formula from earlier.

3. **Questions**

   (a) For $0 \leq \theta \leq \pi$, at what angle is the volume largest? At what angle is the volume smallest? At what angle is the volume changing the fastest?

   (b) What if you change the radius of the bottom plate to something bigger than 1? We have not analyzed this, but you could! Other ideas?