Research: Derivative of Implicit Functions as a Family of Functions

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1 Introduction

I always found the fact that you can't actually map an implicit derivative without a slope field really disappointing, so I set out to find how you could do that, and if it were possible. To study this, I looked principally at the Folium of Descartes, because it's a fascinating and weird curve, and satisfies the main issue with implicit differentiation. The folium is defined as

$$x^3 + y^3 = 3axy$$

with a being a scaling constant.

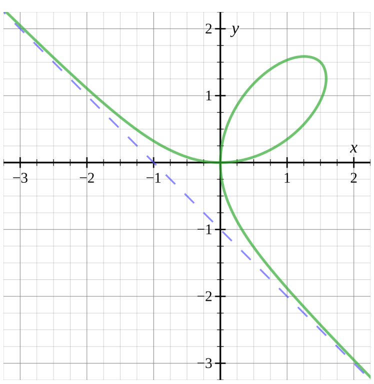

Figure 1: The Folium of Descartes

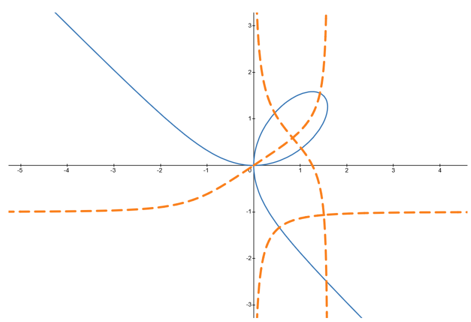

Figure 2: Family of Derivatives for the Folium of Descartes

The goal here is to find the functions that would be a derivative of y with respect to x for all "branches" of the curve, so whenever the function "fails" the vertical line test, there should be multiple derivatives at that point, like how the folium has three y "outputs" per one x "input" for the loop area of the graph. Whenever the function turns around (has a local maxima) on the x axis, the derivative should limit to $\pm\infty$. These are the criteria for what we want, and looking at the function we can draw out what the derivatives should be, but the hard part is finding what they actually are. Here's how I did it:

2 Questions

(a) You can't solve the folium for x or y nicely (try!), so instead we'll look at its parametric form. If you've taken Math 126, you should be able to do this. What is $X(t)$ and $Y(t)$?

(b) Then, we should figure out what the bounds of our function will be. This depends on the x bounds of the loop. This happens whenever $X(t)$ has critical points (whenever $X'(t) = 0$). Find the values for t of those critical points, then plug those values of t back into the parametric equations that you found in part (a). Those are the bounds of the loop, and where our desired derivatives will limit to positive or negative infinity.

(c) Now find $Y'(t)$.

(d) Now think about how we can use this information to get what we want.

The derivative of any function with respect to x is $\frac{dy}{dx}$, and, in a way, we've just found a dy and a dx by finding the derivatives of the parametric equations with respect to t. But if you graph $(X(t), Y(t))$ point-by-point, you'll find that it doesn't flow horizontally like we want. So to make that adjustment the form of our derivatives would be $\frac{Y'(f(x))}{X'(f(x))}$ but what is $f(x)$?

(e) Can you solve $x = X(t)$ for t in terms of x? That's our function $f(x)$! (Consider why...)

It actually ends up being three functions and is a really difficult Cardano's formula problem. But if you can solve those, and then plug that into the composite function we just made, you'll have the desired family of functions that represent the derivative of the Folium of Descartes!

3 Bonus

Can you use this same method to find the family of functions for the derivative of the unit circle? Without just solving the $x^2 + y^2 = 1$ for one variable first? Can you use it for other implicit functions?

Also, this is the start of my capstone, so if you end up messing around with this and thinking of better ways to do it or more applications, tell me about it!