Title:

Approximation Accuracy of Clothoid Arc Length: Comparing Numerical Error and Theoretical Error Bounds.

Concept and Usage:

Taylor series, integral approximations, and theoretical error bounds. Best used in the math126.

Introduction:

The clothoid curve (also called a Fresnel spiral) is widely used in roller coaster tracks and highway ramps due to its property of having curvature that increases linearly with arc length.

To compute the arc length of such curves, engineers ideally evaluate Fresnel integrals, which are computationally expensive. Instead, many use finite-order Taylor polynomials to approximate the curve — a practical shortcut that introduces approximation error.

This project compares:

Numerical Error：

The actual difference between the approximated arc length and the “true” arc length computed via numerical integration

Theoretical Error Bound：

An upper limit derived from Taylor’s remainder term, estimating how large the error could possibly be

We explore how closely this theoretical bound predicts the true error and how both behave under different approximation orders.

Question:

1. What is the difference between the numerical error and the theoretical error bound when approximating clothoid arc length using Taylor series?

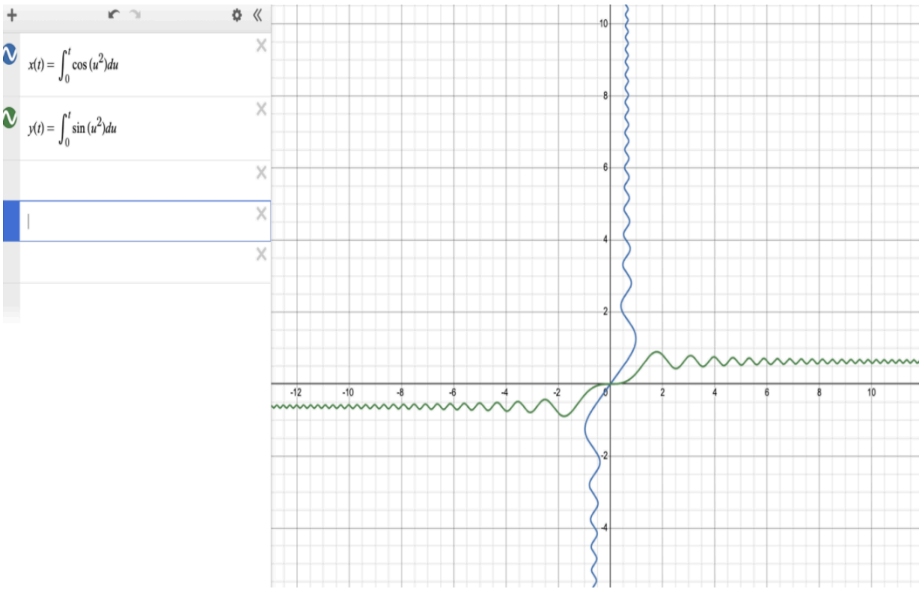
2. How does the numerical error behave as the Taylor approximation order increases?

3. How does the theoretical error bound change with Taylor order and arc length?

4. How can these insights guide engineering practice?

Visual

Fresnel integral



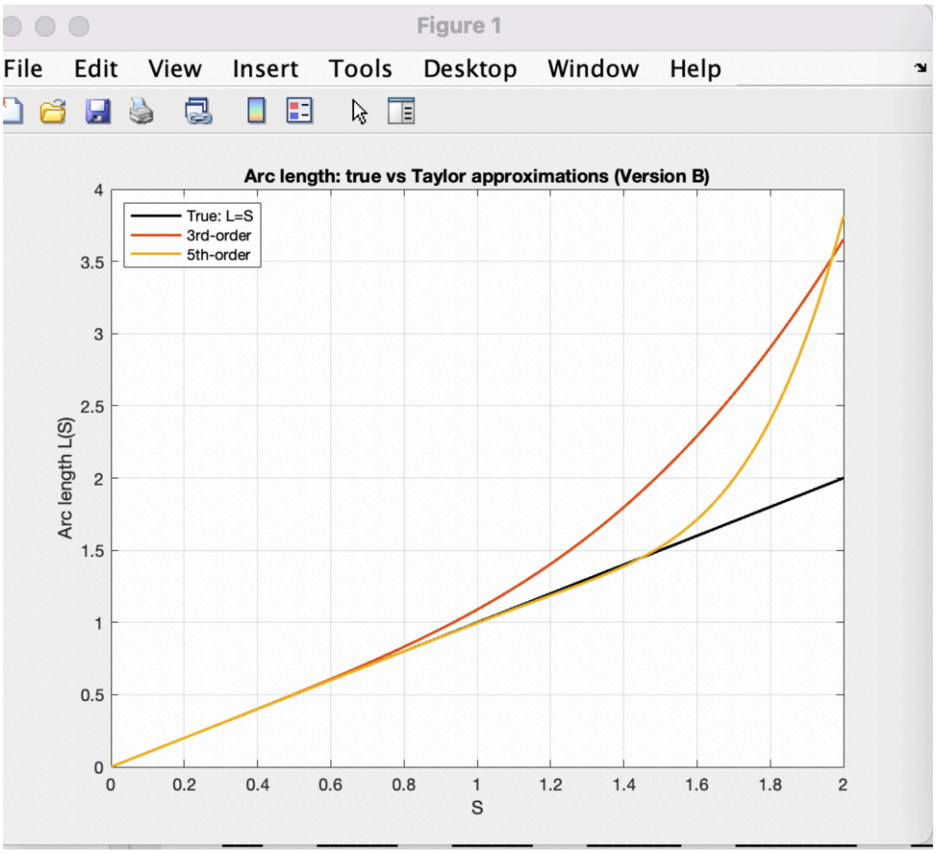


Figure 1
File Edit View Insert Tools Desktop Window Help
Arc length: true vs Taylor approximations (Version B)
True: L=S
3rd-order
5th-order
Arc length L(S)
S
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.5
1.5
2.5
3
3.5
4

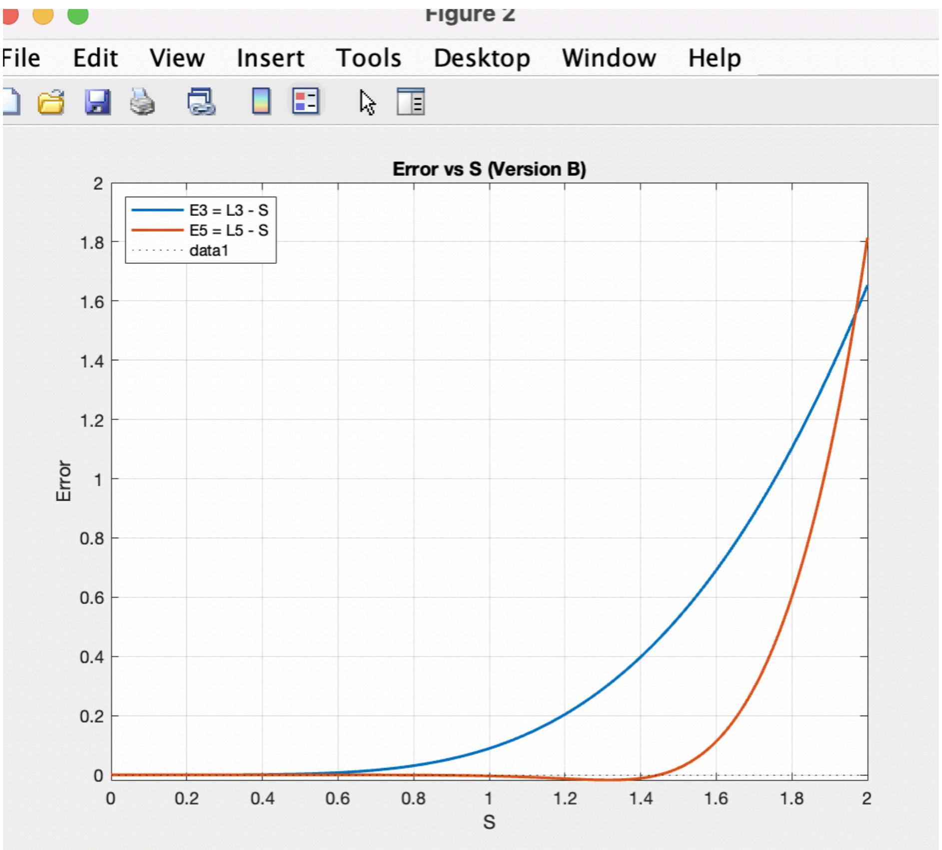
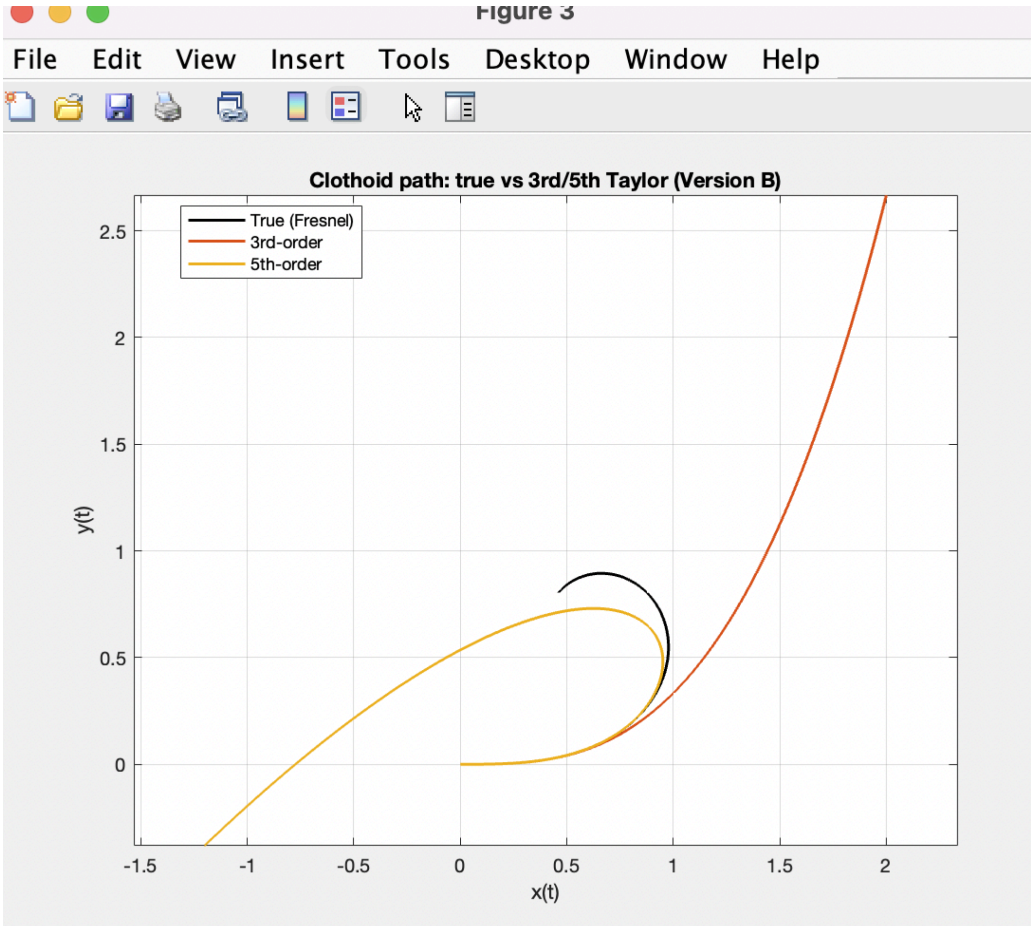
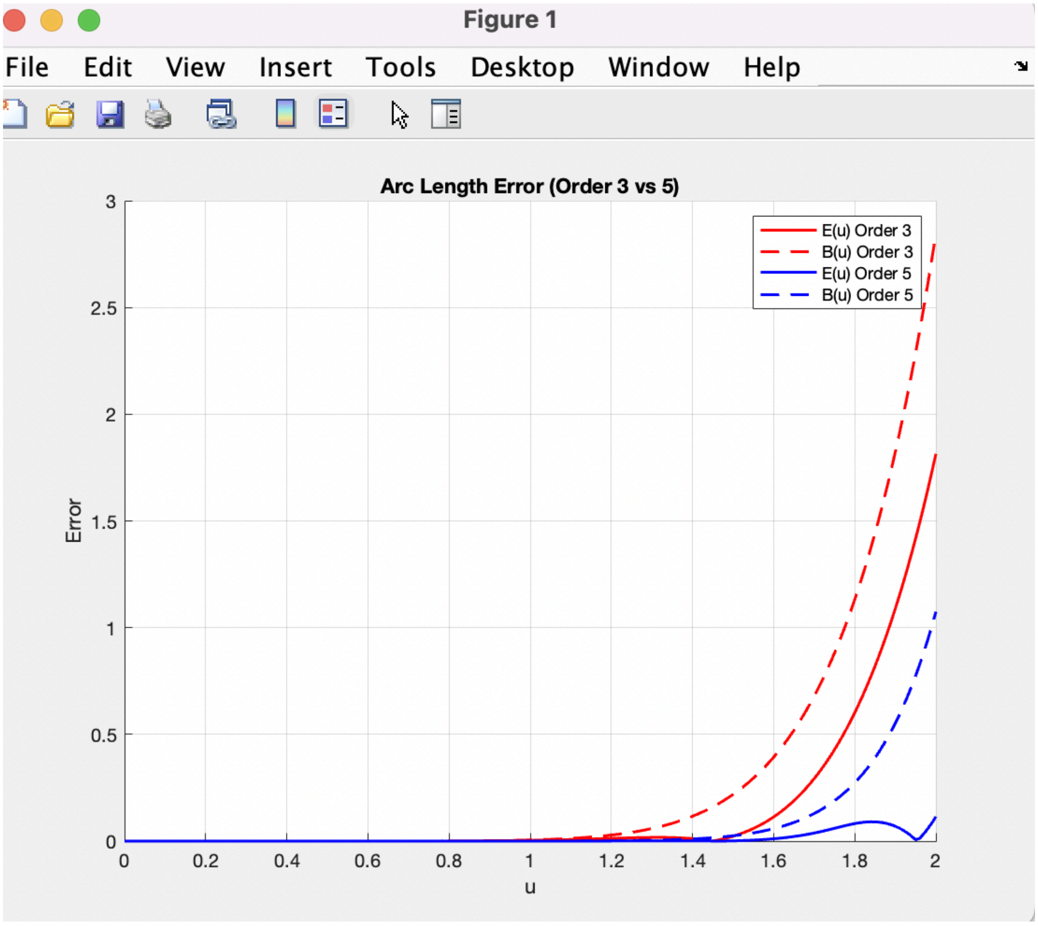


Figure 2
File Edit View Insert Tools Desktop Window Help
Error vs S (Version B)
E3 = L3 - S
E5 = L5 - S
data1
Error
S
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

This figure shows that the 5th-order Taylor approximation is consistently more accurate than the 3rd-order, and that approximation error increases as the arc length grows.





Numerical errors are far smaller than theoretical error bounds, higher-order Taylor expansions (like 5th order) greatly improve accuracy, and the theoretical bound offers a conservative guarantee that ensures safety in engineering applications。

Reflection and further research:"Why does the theoretical error bound for the 5th-order approximation suddenly spike near u=2, while the actual numerical error remains smooth? Is this purely due to the behavior of higher-order derivatives in the Taylor remainder, or could there be alternative ways to construct a tighter, more stable bound?"

Details on the set-up:

Definition [edit]

The Fresnel integrals admit the following Maclaurin series that converge for all x:

$$S(x) = \int_0^x \sin(t^2)\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(2n+1)!(4n+3)},$$

$$C(x) = \int_0^x \cos(t^2)\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(2n)!(4n+1)}.$$

Here u is a parameter proportional to arc length, so the exact arc length from 0 to x。simply s=x.To make the problem more tractable, we approximate the Fresnel integrals by Taylor expansions cos(t^2) and sin(t^2).

$$\sin(t^2) = t^2 - \frac{t^6}{3!} + \cdots, \quad \Rightarrow \quad S(s) = \int_0^s \sin(t^2)dt \approx \int_0^s t^2 dt = \frac{s^3}{3}$$

$$\cos(t^2) = 1 - \frac{t^4}{2!} + \cdots, \quad \Rightarrow \quad C(s) \approx \int_0^s 1\,dt = s$$

Substituting these into the integrals, we obtain polynomial approximations for x(u) and y(u)。

These give us approximate parametric curves that are easier to compute than the Fresnel integrals.

The arc length of the approximate curve is then:

$$L(u) \approx \int_0^u \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt,$$

where x′(t),y′(t) come from the truncated Taylor expansions.

By comparing the 3rd-order and 5th-order approximations with the exact length s=u, we can quantify the error introduced by truncating the series.

Numerical error:

$$E_{\text{num}}(u) = |L(u) - u|$$

Theoretical Error Bound

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

$$s(u) = \int_0^u \left((\cos(t^2))^2 + (\sin(t^2))^2\right) dt = u$$

$$L(u) = \int_0^u \left((P_n^{(\cos)}(t))^2 + (Q_n^{(\sin)}(t))^2\right) dt$$

$$E(u) = |L(u) - s(u)| = \left|\int_0^u \left((P_n^{(\cos)}(t))^2 + (Q_n^{(\sin)}(t))^2 - 1\right) dt\right|$$

$$\leq \int_0^u \left(|P_n^{(\cos)}(t) - \cos(t^2)| + |Q_n^{(\sin)}(t) - \sin(t^2)|\right) dt$$

$$\leq \int_0^u \left(|R_n^{(\cos)}(t)| + |R_n^{(\sin)}(t)|\right) dt$$

$$E(u) \leq B_n(u)$$

$$B_n(u) = \int_0^u \left(|R_n^{(\cos)}(t)| + |R_n^{(\sin)}(t)|\right) dt$$