The Apollonius Circle Problem: Tangent Circles to Three Circles

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Concept and Usage

The Apollonius problem asks: given three circles in a plane, find a fourth circle that is tangent to all three. The goal of this project is to derive a general formula for such a tangent circle, which can then be used in geometry, design, and circle-packing applications.

Introduction

Given three circles with centers (x_i, y_i) and radii r_i, $i = 1, 2, 3$, the tangent circle with center (x, y) and radius R satisfies the tangency equations:

$$(x - x_i)^2 + (y - y_i)^2 = (R - r_i)^2, \quad i = 1, 2, 3.$$

After algebraic manipulation, these can be transformed into a system of linear equations:

$$A_1 x + B_1 y = C_1 R + D_1, \quad A_2 x + B_2 y = C_2 R + D_2,$$

where

$$\begin{aligned} A_i &= x_1 - x_j, \quad B_i = y_1 - y_j, \\ C_i &= r_1 - r_j, \quad D_i = \tfrac{1}{2}(r_j^2 - r_1^2 + x_1^2 - x_j^2 + y_1^2 - y_j^2), \end{aligned}$$

for $i = 1, 2$ and $j = i + 1$ the other index. Solving this system for (x, y) in terms of R gives:

$$y(R) = \frac{(A_2 C_1 - A_1 C_2) R}{A_2 B_1 - A_1 B_2}, \quad x(R) = \frac{C_1 R + D_1 - B_1 y(R)}{A_1}.$$

Substituting back into any of the tangency equations allows solving for the radius R of the tangent circle. Moreover, our assumptions for this solution is that radii for all 3 circle equations has to be equal. Consider three circles:

$$(x - 0)^2 + (y - 0)^2 = 1^2, \quad (x - 4)^2 + (y - 0)^2 = 1^2, \quad (x - 2)^2 + (y - 3)^2 = 1^2.$$

Using the formulas above, we find a tangent circle with center and radius approximately:

$$(x, y) \approx (2, 0.83), \quad R \approx 3.17, -1.17.$$

Extension to Spheres

The method for tangent circles can be extended to three-dimensional space to find a sphere tangent to four given spheres. Let the four spheres be:

$$\begin{aligned}
&(x-1)^2 + y^2 + z^2 = 1,\\
&(x+1)^2 + y^2 + z^2 = 1,\\
&x^2 + (y-\sqrt{3})^2 + z^2 = 1,\\
&x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 + \left(z - 2\sqrt{\frac{2}{3}}\right)^2 = 1.
\end{aligned}$$

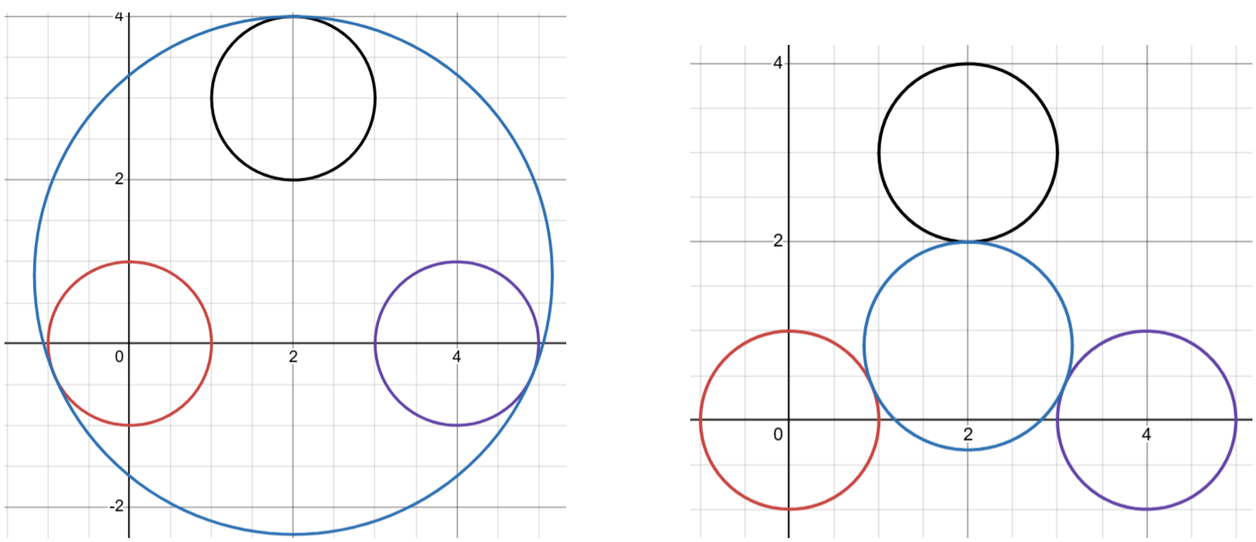
Solving for a sphere tangent to all four gives an inner and outer tangent sphere:

$$\text{Inner Sphere: } x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 + \left(z - \frac{1}{\sqrt{6}}\right)^2 = \left(\sqrt{\frac{3}{2}} - 1\right)^2,$$

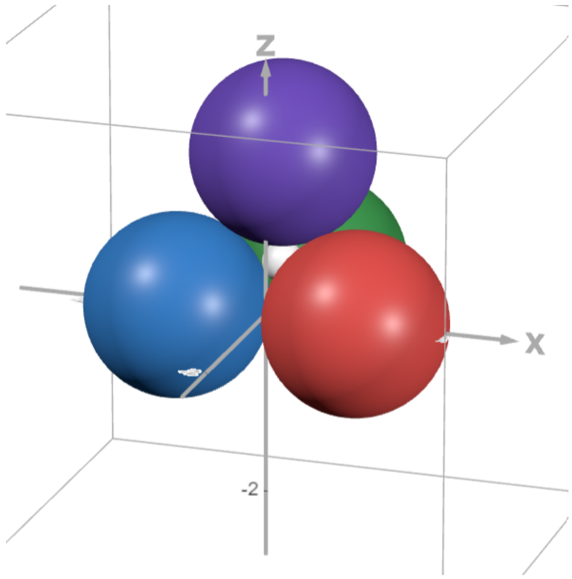
$$\text{Outer Sphere: } x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 + \left(z - \frac{1}{\sqrt{6}}\right)^2 = \left(\sqrt{\frac{3}{2}} + 1\right)^2.$$

This illustrates how the tangent formulas generalize from circles to spheres, including inner and outer solutions.

Visuals



Left: Outer circle to three given circles. Right: Inner circle to the same three circles.



White inner tangent sphere to four given spheres.

Questions

1. Is there a general formula for finding a tangent sphere given four or more spheres?
2. Can the Apollonius problem for circles be generalized to n circles, and can we derive a formula for the tangent circle in that case?
3. Similarly, can we extend the sphere problem to n spheres and find a formula for a tangent sphere?
4. While the visuals show only the inner and outer tangent circles/spheres, there are actually 8 possible circles and 16 possible spheres—can we visualize or categorize all solutions in general?