Air Resistance and Terminal Velocity
by Dr. Andy Loveless

Concepts and usage: Differential Equations. Could be asked in last two weeks of Math 125.

Introduction: Air resistance increases as an object travels at a faster velocity through the air. For this reason, an object that falls from an airplane will eventually get to a velocity where the force due to gravity and the force due to air resistance is the same. From that moment on the object will fall at a constant velocity, which we call **terminal velocity**.

Assume we have an object of mass m dropped from an initial height of h meters. Let $y(t)$ be the height above the ground after t seconds and let $v(t) = y'(t)$ be the velocity of the object, which will be negative in this set-up. Newton's second law says the sum of all the forces on an object equals mass times acceleration (which is $v'(t)$). For the questions below we will explore the forces due to gravity $F_g = -mg$ and air resistance F_a for this dropping object. Thus, the general set up is $m\dfrac{dv}{dt} = F_g + F_a$.

Assume $g = 9.8$ m/s^2 and a ball is dropped from 300 meters (the top of a skyscraper) in all the questions below.

1. *Only gravity, $F_a = 0$*: We have $m\frac{dv}{dt} = -mg$.
   (a) Find the solution for $v(t)$ and $y(t)$.
   (b) Give the time it will take for the object to reach the ground.
   (c) Find the velocity at the time it hits the ground.
2. Two common models for air resistance in the situation of a dropped ball and their corresponding terminal velocity formulas are given:
   - *Air resistance proportional to velocity, $F_a = -k_1 v$*:
     - $m\frac{dv}{dt} = -mg - k_1 v$, where k_1 is a positive constant. Terminal velocity $= -\frac{mg}{k_1}$.
   - *Air resistance proportional to the square of velocity, $F_a = k_2 v^2$*:
     - $m\frac{dv}{dt} = -mg + k_2 v^2$, where k_2 is a positive constant. Terminal velocity $= -\sqrt{\frac{mg}{k_2}}$.
3. Comparing models: Choose one of the sports balls on the next page and compute k_1 and k_2 based on their terminal velocities. Again, assume $g = 9.8$ m/s^2 and the ball is dropped from 300 meters. *You may use an online solver to help give numerical answers to the questions below.*
   (a) Using k_1, solve $m\frac{dv}{dt} = -mg - k_1 v$ for $v(t)$ and find time it will take for the object to hit the ground given this model.
   (b) Using k_2, solve $m\frac{dv}{dt} = -mg + k_2 v^2$ for $v(t)$ and find time it will take for the object to hit the ground given this model.
   (c) For all three models compare the time and velocities when your ball hits the ground and summarize in a table.

Finding Terminal Velocity At the terminal velocity we get that acceleration is zero, so we can solve for that velocity directly in the two models on the previous page by replacing $\frac{dv}{dt}$ by 0 and solving for v. We call this an 'equilibrium point' of the differential equation and the study of their behavior is one of the tools you'll study in a differential equation course. Here is what you get in the case of our two models.

- If $\frac{dv}{dt} = 0$ in $m\frac{dv}{dt} = -mg - k_1 v$, then we solve to get $v_{term} = -\frac{mg}{k_1}$
- If $\frac{dv}{dt} = 0$ in $m\frac{dv}{dt} = -mg + k_2 v^2$, then we solve to get $v_{term} = -\sqrt{\frac{mg}{k_2}}$.

Here are a collection of terminal velocities given by an internet search which you will use to find k.

- Baseball ≈ -95 mph (-42.5 m/s and it is about 0.145 kg).
- Golf ball ≈ -72 mph (-32 m/s and it is about 0.05 kg)
- Tennis ball ≈ -48 mph (-21.5 m/s and it about 0.05 kg)
- Basketball ≈ -47 mph (-21 m/s and it is about 0.62 kg)
- Soccer ball ≈ -45 mph (-20 m/s and it is about 0.45 kg)
- Ping Pong Ball ≈ -20 mph (-9 m/s and it is about 0.0027 kg)

For example, for a tennis ball we have $v = -21.5$, $g = 9.8$ and $m = 0.05$.

- Plugging these values into $v = -\frac{mg}{k_1}$, we get $k_1 = -vmg = -(-21.5)(0.05)(9.8) = 10.535$.
- Plugging these values into $v = -\frac{mg}{k_2}$, we get $k_2 = -v^2 mg = (-21.5)^2(0.05)(9.8) = 226.5025$.

Ideas for Further Research

1. Look at different exponents for the velocity in $m\frac{dv}{dt} = -mg \pm kv^{\alpha}$, where α is a constant between 1 and 2. For a golf ball, I once read they used $\alpha = 1.1$, but it appears that $\alpha = 2$ is the most commonly used in my readings about air resistance.
2. If you want to study motion in 2 or 3-dimensions, then the general set up is typically given in vector form $m\mathbf{r}''(\mathbf{t}) = \mathbf{F}_g + \mathbf{F}_a$ which means you need to figure the components of force in the x, y and z directions. It would be interesting to do a full study of the equations used in these cases...
   (a) If the only force is gravity, what is the differential equation for x'', y'' and z''?
   (b) If we use the first model or second model for air resistance, what is the differential equation for x'', y'' and z''?
3. If you wanted to account for spin, then you need to also include the "magnus force" and the set up looks like $m\mathbf{r}''(\mathbf{t}) = \mathbf{F}_g + \mathbf{F}_a + \mathbf{F}_m$. It would be interesting to explore the component-wise differential equations in this case. These equations are messier, but would be nice to encode in Desmos in a way that we can explore, visualize and animate.