## 15.9: Changing Variables in Double and Triple Integrals

**TRANSFORMATION BASICS**: A **transformation** $T(u, v) = (x, y)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ (2D to 2D) or $T(u, v, w) = (x, y, z)$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ (3D to 3D) is a function that define the 'new' points from the 'old' points.
A 2D transformation can be written as: $x = g(u, v)$, $y = h(u, v)$.
A 3D transformation can be written as: $x = g(u, v, w)$, $y = h(u, v, w)$, $z = k(u, v, w)$.

1. A $\mathbf{C}^1$ **transformation** is a transformation where the functions g, h, and k all have continuous first partial derivatives (basically we can differentiate and the derivatives are nicely behaved, which accounts for most situations we encounter in this class).

2. Given a set of points $S$ in the $uv$-plane, the **image of S** under the transformation $T$ is the set of all points $R$ in the $xy$-plane that are output from $S$. A transformation is said to be *one-to-one* if no two points map to the same image point. If a transformation is one-to-one between two sets, then we can discuss the **inverse transformation** $T^{-1}$ that maps points back from the $R$ to $S$.

3. **The definitions above may seem like 'mathematician speak' that doesn't matter, but, in fact, you really do need to visualize and consider these images when you change the bounds in a double/triple integral**.

**FINDING THE IMAGE**: Assume we are given a double integral (with bounds) and we want to change the variable. Suppose the given integration region in the $xy$-plane is $D$ and the desired transformation is $x = g(u, v)$ and $y = h(u, v)$. We need to find the image of $D$ in the $uv$-plane in order to find the new region of integration. Here are some methods to help you do that:

1. First, make sure you have a good description of the region $D$ in the $xy$-plane. Find the equations for all boundaries/sides of the region $D$. Find all 'corners'.

2. One at a time, for each side of $D$, determine the image of the side. You do this by replacing $x = g(u, v)$ and $y = h(u, v)$ in the equation of the side. Now you have an equation in terms of $u$ and $v$ that describes the image in the $uv$-plane for that side. Also, solve for $u$ and $v$ corresponding to the $(x, y)$ points of the corners of $D$.

3. Graph each of the new sides and corners in the $uv$-plane. You should have the new region $R$ over which you will be integrating in the new variables.

This can be time consuming in general, but we often deal in basic shapes (rectangles, triangles, circles) in which this isn't too bad. This is part of the decision making process when we are trying to decide if a change of variable is worthwhile.

**NOW FOR THE CALCULUS PART**.

1. *Motivation*: In Calculus II, we learned how to 'reverse' the chain rule in an organized way by doing a change in variable (often referred to as a $u$-substitution). The rule noted that if $F(x) + C = \displaystyle\int f(x)dx$ and $x = g(u)$, then $F(g(u)) + C = \displaystyle\int f(g(u))g'(u)du$. You have used this many times in Math 125, 126, and in this class so far. In Math 125, we write $x = g(u)$ and, the differential, $dx = g'(u)du$, then this rule is easy to remember. Noting $g'(u) = \frac{dx}{du}$, the rule can also be written as:
$$\int_a^b f(x)dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du.$$
where $x = a$ corresponds to $u = c$ and $x = b$ corresponds to $u = d$ (we have to change the bounds to match the variable). So $dx$ becomes $\frac{dx}{du}\,du$. Now we generalize to double and triple integrals.

2. *Double integrals, what happens to $dA$:* If we want to integrate with respect to new variables $u$ and $v$, we needed to figure out how the area of a small rectangle the $uv$-plane would relate to the area of its image in the $xy$-plane. We found the area of the image can be approximated by a parallelogram with sides $\Delta u \mathbf{r}_u$ and $\Delta v \mathbf{r}_v$, where $\mathbf{r}_u$ and $\mathbf{r}_v$ were the tangent vectors of the image of the sides of the rectangle (see the book for an illustration). From Math 126, the area of the parallelogram is

$$\text{Area of image } \approx |(\Delta u \mathbf{r}_u) \times (\Delta v \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$$

So we define the **Jacobian** of the transformation $T$ given by $x = g(u,v)$ and $y = h(u,v)$, by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

We found the $\mathbf{r}_u \times \mathbf{r}_v = \langle 0, 0, \frac{\partial(x,y)}{\partial(u,v)} \rangle$. Using this notation, we had then argued that a small rectangle with sides $\Delta u$ and $\Delta v$ in the $uv$-plane is transformed to a region in the $xy$-plane with area: $\Delta A \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$.

Assuming $T$ is a sufficiently nice (a $C^1$ transformation and one-to-one), $f$ is continuous, and $S$ is the image of $R$ under $T$, then we have

$$\iint\limits_R f(x,y)\, dA = \iint\limits_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv.$$

To use this:

(a) Compute the Jacobian.

(b) Find the image of $R$.

(c) Replace $x = x(u,v)$ and $y = u(u,v)$.

(d) Replace $dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$.

3. *Triple integrals, what happens to $dV$:* The discussion is almost identical. First we must define the **Jacobian** for three dimension which is:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Assuming $T$ is a sufficiently nice (a $C^1$ transformation and one-to-one), $f$ is continuous, and $S$ is the image of $R$ under $T$, then we have

$$\iiint\limits_R f(x,y,z)\, dV = \iiint\limits_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw.$$