15.3, 15.4, 15.5 and 15.6 Review

My reviews and review sheets are not meant to be your only form of studying. It is vital to your success on the exams that you carefully go through and understand ALL the homework problems and understand them. Hopefully this review sheet will remind you of some of the key ideas of these sections.

15.5: Double Integral Applications

1. First note that:
$$A(D) = \iint_D 1 \, dA = \text{Area of D.}$$
$$\frac{1}{A(D)} \iint_D f(x,y) \, dA = \text{Average value of } f(x,y) \text{ on D.}$$

2. Assume a region D describes a thin plate (or lamina). If $\rho(x,y)$ is the density (mass per area) at the point (x,y), then we can find the center of mass $(\bar{x}, \bar{y})$ as follows:
$$\begin{aligned} m &= \iint_D \rho(x,y) \, dA = \text{total mass of the plate.} \\ M_y &= \iint_D x\rho(x,y) \, dA = \text{moment } \textit{about} \text{ the } y\text{-axis, and} \quad \bar{x} = \frac{M_y}{m}. \\ M_x &= \iint_D y\rho(x,y) \, dA = \text{moment } \textit{about} \text{ the } x\text{-axis, and} \quad \bar{y} = \frac{M_x}{m}. \end{aligned}$$

3. Similarly if $\sigma(x,y)$ is the electric charge density (charge per area) at the point (x,y), then
$$Q = \iint_D \sigma(x,y) \, dA = \text{total charge on the plate.}$$

4. The **moment of inertia** of a single particle of mass m that is r units from the axis is defined to be mr^2. For a thin place with density $\rho(x,y)$ we can compute the moments of inertia relative to the axes as follows:
$$\begin{aligned} I_y &= \iint_D x^2\rho(x,y) \, dA = \text{moment of inertia about the } y\text{-axis.} \\ I_x &= \iint_D y^2\rho(x,y) \, dA = \text{moment of inertia about the } x\text{-axis.} \\ I_o &= \iint_D (x^2+y^2)\rho(x,y) \, dA = I_x + I_y = \text{moment about the origin.} \end{aligned}$$

   The moment of inertia gives information about resistance to change in rotation. If we increase mass or how far the mass is from the origin, we increase moment of inertia (and make it harder to rotate).

5. This is not a physics class, so we don't go into depth into the implication of these applications in this class. Ultimately, this section gives you an opportunity to practice double integrals in some common situations where they are used. So you should know the integrals, know basically what they represent, and know how to use them. For for in depth discussion, you'll have to wait for your physics and other science classes.

15.6: Triple Integrals

1. Given a real valued three variable function $w = f(x,y,z)$ and a solid region E, we define the **triple integral** as $\displaystyle\iiint_E f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$

2. Given a solid region E, there are 6 different ways to set up our integral (one for each ordering of dx, dy and dz). So that gives a lot of options. But ultimately we group these into three major cases. **In all cases and in practice, we start by deciding what variable we want for the inner most integral.**

   (a) **Type I (TOP/BOTTOM or dz inside):**
   If $u_1(x,y) \le z \le u_2(x,y)$ and D is the projection of the solid onto the xy-plane, then
   $$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right)\,dA.$$

   (b) **Type II (BACK/FRONT or dx inside):**
   If $u_1(y,z) \le x \le u_2(y,z)$ and D is the projection of the solid onto the yz-plane, then
   $$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right)\,dA.$$

   (c) **Type III (LEFT/RIGHT or dy inside):**
   If $u_1(x,z) \le y \le u_2(x,z)$ and D is the projection of the solid onto the xz-plane, then
   $$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right)\,dA.$$

3. So the process is:

   (a) Make a variable choice and solve each equation to get that variable by itself. (Let's say you choose the variable z)

   (b) Then draw the projection onto the other two variables. (The xy-plane if you chose z in the previous step).

      i. If the surface actually cross the plane, then you can set the 'inner' variable choice to zero ($z = 0$) in the surface to find the curve in which the surface intersect the plane.

      ii. If there are two surfaces you may need to find the curve of intersection (meaning you have $z =$ surface 1 function and $z =$ surface 2 function, you set surface 1 function $= z$ $=$ surface 2 function and simplify to get the curve of intersection).

      iii. Graph all these in the other two variables.

   (c) Then use the techniques of 15.3 and 15.4 to describe D with inequalities.

4. Applications:

$$\iiint_E 1\, dV = \text{Volume of E.}$$

$$\frac{1}{V(E)} \iiint_E f(x,y,z)\, dV = \text{Average value of } f(x,y,z) \text{ on E.}$$

$$\begin{aligned}
m &= \iiint_E \rho(x,y,z)\, dV = \text{total mass.} \\
M_{yz} &= \iiint_E x\rho(x,y,z)\, dV = \text{moment } \textit{about} \text{ the } yz\text{-plane, and} \quad \bar{x} = \frac{M_{yz}}{m}. \\
M_{xz} &= \iiint_E y\rho(x,y,z)\, dV = \text{moment } \textit{about} \text{ the } xz\text{-plane, and} \quad \bar{y} = \frac{M_{xz}}{m}. \\
M_{xy} &= \iiint_E z\rho(x,y,z)\, dV = \text{moment } \textit{about} \text{ the } xy\text{-plane, and} \quad \bar{z} = \frac{M_{xy}}{m}. \\
Q &= \iiint_E \sigma(x,y,z)\, dV = \text{total charge.}
\end{aligned}$$

15.3/15.4: Some Notes and Ramblings on 'visualizing' regions

1. In some problems and situation, a solid is described as bounded by several three dimensions surfaces. You had a few problems in the homework that said find the volume of the region bounded by BLAH, BLAH, BLAH. Here are some notes that may help you in those situations.

   (a) Determine your surface. Unless I missed one, I believe every problem you did from 15.3 and 15.4 ultimately was bounded above by some surface $z = f(x,y)$ and below by a region on the xy-plane. So look for all the equations that have z's in them and solve for z in each one.

   (b) Once you have the surface, find the region on the xy-plane. To get the projection of the surface $z = f(x,y)$ onto the xy-plane, set $z = 0$. So you can plot $0 = f(x,y)$ in the xy-plane along with any other boundaries given in the problem.

   (c) Now you should be able to give inequalities for the region and you should be set to go.

2. It does not have to always be the case that it is bounded above by a function $z = f(x,y)$, but it was done in 15.3 and 15.4 to keep it simple (when we do triple integrals we will focus on other possibilities like being bounded to the 'right' by $y = f(x,z)$ and to the 'left' by some region on the xz-plane, or in the 'front' by $x = f(y,z)$ and in the 'back' by some region on the yz-plane). If things were more complicated then we may have to spend more time trying to get a good 3D picture, but this is rarely necessary.

3. Another good question is: How can I get a sense of the 3D surface if I'm not good at drawing 3D pictures? ANSWER: Draw a contour map. Here is a reminder of how to do that for $z = f(x,y)$.

   (a) Simplify $0 = f(x,y)$, draw the resulting curve in the xy-plane and label the curve '$z = 0$'.

   (b) Simplify $1 = f(x,y)$, draw the resulting curve in the xy-plane and label the curve '$z = 1$'. And so on ...

   (c) You can also draw such a map from the sides as well by fixing $x = 0$, $x = 1$, etc (then doing $y = 0$, $y = 1$, etc).