1. (10 pts) Consider the vector field $\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} + zy\,\mathbf{j} + yx^3\,\mathbf{k}$ on $\mathbb{R}^3$.

(a) (6 pts) Compute the following:

i. $\operatorname{curl}\mathbf{F}$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & zy & yx^3 \end{vmatrix} = (x^3 - y)\vec{i} - (3yx^2 - 0)\vec{j} + (0 - x^2)\vec{k}$$

$$= \boxed{(x^3 - y)\vec{i} - 3yx^2\vec{j} - x^2\vec{k}}$$

ii. $\nabla(\operatorname{div}\mathbf{F})$.

$\operatorname{div}\vec{F} = 2xy + z + 0$

$\nabla(\operatorname{div}\vec{F}) = \boxed{\langle 2y, 2x, 1\rangle}$

iii. $\operatorname{div}(\operatorname{curl}\mathbf{F})$. $= \boxed{0}$ ← Always

(b) (4 pts) Give a short one sentence answer to each of the two questions below:

i. What can you conclude for a vector field where $\operatorname{curl}\mathbf{F} \neq \mathbf{0}$?

$\vec{F}$ is not conservative.

ii. What can you conclude for a vector field where $\operatorname{div}\mathbf{F} \neq 0$?

$\vec{F}$ is not the curl of another vector field.

2. (8 pts)

(a) (4 pts) Give a parameterization for the part of surface $y^2 + z^2 - x = 3$ with $0 \leq x \leq 1$. Include bounds on the parameters.

BECAUSE OF THE BOUNDS, EASIEST TO USE

$$\boxed{\begin{aligned} y &= v\cos(u) \\ z &= v\sin(u) \\ x &= v^2 - 3 \end{aligned}}$$

$$\Rightarrow \underbrace{y^2 + z^2}_{} - x = 3$$

$$v^2 - x = 3$$

$$\Rightarrow x = v^2 - 3$$

v positive

$0 \leq u \leq 2\pi$

BOUNDS

$$0 \leq x \leq 1$$
$$0 \leq v^2 - 3 \leq 1$$
$$3 \leq v^2 \leq 4$$
$$\boxed{\begin{aligned} \sqrt{3} \leq v &\leq 4 \\ 0 \leq u &\leq 2\pi \end{aligned}}$$

(b) (4 pts) You are told that $x = x(t)$, $y = y(t)$, and $z = z(t)$ is the parameterization for the motion of some particle along the curve C which is on the surface $z = x^2 + \sin(y) - xy^2$. If $x(1) = 2$, $y(1) = 0$, $x'(1) = 3$, and $y'(1) = -5$, then what is the value of $z'(1)$? That is, find $\frac{dz}{dt}$ at $t = 1$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x - y^2)\frac{dx}{dt} + (\cos(y) - 2xy)\frac{dy}{dt}$$

$$\left.\frac{dz}{dt}\right|_{t=1} = (2(2) - (0)^2)(3) + (\cos(0) - 2(2)(0))(-5)$$

$$= 12 - 5 = \boxed{7}$$

3. (9 pts) Consider the vector field $\mathbf{F}(x,y,z) = (-z\sin(x) + y^2)\mathbf{i} + (2xy + e^{z^2})\mathbf{j} + (\cos(x) + 2yze^{z^2})\mathbf{k}$ on $\mathbf{R}^3$. You are told that the vector field is conservative!

(a) (6 pts) Find a function $f(x,y,z)$ such that $\nabla f = \mathbf{F}$.

$f_x(x,y,z) \stackrel{?}{=} -z\sin(x) + y^2 \Rightarrow f(x,y,z) = z\cos(x) + xy^2 + g(y,z)$

$f_y(x,y,z) \stackrel{?}{=} 2xy + e^{z^2} \Rightarrow 0 + 2xy + g_y(y,z) \stackrel{?}{=} 2xy + e^{z^2}$

$\Rightarrow g_y(y,z) = e^{z^2}$

$g(y,z) = ye^{z^2} + h(z)$

$f(x,y,z) = z\cos(x) + xy^2 + ye^{z^2} + h(z)$

$f_z(x,y,z) \stackrel{?}{=} \cos(x) + 2yze^{z^2} \Rightarrow \cos(x) + 0 + 2yze^{z^2} + h'(z) \stackrel{?}{=} \cos(x) + 2yze^{z^2}$

$\Rightarrow h'(z) = 0$

$h(z) = C \leftarrow$ a constant

GENERAL ANSWER:

$$\boxed{f(x,y,z) = z\cos(x) + xy^2 + ye^{z^2} + C}$$

(b) (3 pts) Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ over the curve, C, given by $\mathbf{r}(t) = \langle \pi t, 3 - 3t^4, \sin(\pi t) + 5t\rangle$ for $0 \le t \le 1$. (Please think about your options here.)

START POINT: A $\vec{r}(0) = \langle 0, 3, 0\rangle$ $\quad A = (0,3,0)$

END POINT: B $\vec{r}(1) = \langle \pi, 0, 5\rangle$ $\quad B = (\pi, 0, 5)$

$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A) = f(\pi,0,5) - f(0,3,0)$

$= \left[(5)\cos(\pi) + (\pi)(0)^2 + (0)e^{(5)^2}\right] - \left[(0)\cos(0) + (0)(3)^2 + 3e^{(0)^2}\right]$

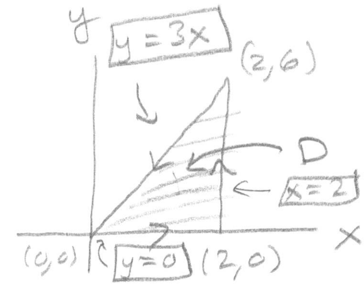
$= -5 - 3$

$= \boxed{-8}$

4. (8 pts) Use Green's Theorem to evaluate

$$\oint_C \sin(x^3)\, dx + 4x^2 y\, dy$$

where C is the triangle with vertices $(0,0)$, $(2,0)$, and $(2,6)$.



$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^2 \int_0^{3x} (8xy - 0)\, dy\, dx$$

$$= \int_0^2 4xy^2 \Big|_0^{3x} dx$$

$$= \int_0^2 36x^3\, dx$$

$$= \frac{36}{4} x^4 \Big|_0^2 = 9 \cdot 2^4 = 9 \cdot 16 = \boxed{144}$$

5. (5 pts) Assume the temperature at each point on the xy-plane is given by

$$T(x,y) = \frac{1}{3}x^2 y + 5\sqrt{x^2+y^2} \quad \text{degrees Celcius,}$$

where x and y are in feet. Find the directional derivative of $T(x,y)$ at the point $(3,4)$ in the direction of $\langle -1, 2\rangle$. Give the units for your answer.

$$\nabla T(x,y) = \left\langle \frac{2}{3}xy + \frac{5x}{\sqrt{x^2+y^2}},\ \frac{1}{3}x^2 + \frac{5y}{\sqrt{x^2+y^2}} \right\rangle$$

$$\nabla T(3,4) = \left\langle 8 + \frac{5\cdot 3}{5},\ \frac{1}{3}(3)^2 + \frac{5\cdot 4}{5} \right\rangle = \langle 11, 7 \rangle$$

UNIT DIRECTION VECTOR $= \vec{u} = \frac{1}{\sqrt{5}} \langle -1, 2 \rangle$

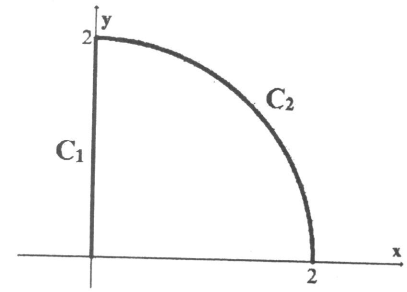
$$D_{\vec{u}} T(3,4) = \nabla T(3,4) \cdot \frac{1}{\sqrt{5}} \langle -1, 2 \rangle$$

$$= \frac{1}{\sqrt{5}} (-11 + 14) = \boxed{\frac{3}{\sqrt{5}} \quad \frac{°C}{ft}}$$

6. (10 pts) Assume, again, the temperature at each point on the xy-plane is given by $T(x,y) = \frac{1}{3}x^2y + 5\sqrt{x^2+y^2}$ degrees Celcius. You are told that the average temperature along a curve C is given by $\frac{1}{L}\int_C T(x,y)\,ds$, where L is the total length of C.

Let C be the curve consisting of a straight line segment from the origin to $(0,2)$, then one quarter of the circle $x^2+y^2=4$ from $(0,2)$ to $(2,0)$. Compute the average temperature along C. That is, compute $\frac{1}{L}\int_C T(x,y)\,ds$.

(Hint: Parameterize!)



NOTE: $L = \underbrace{2}_{\text{length of } C_1} + \underbrace{\pi}_{\text{length of } C_2\ \frac{1}{4}(\pi r)}$

C_1: $x = 0,\ y = 2t, \quad 0 \le t \le 1$

$x' = 0,\ y' = 2$

$ds = \sqrt{0^2 + 2^2}\,dt = 2\,dt$

$$\int_{C_1} T(x,y)\,ds = \int_0^1 \left(\tfrac{1}{3}(0)^2(2t) + 5\sqrt{0^2+(2t)^2}\right) 2\,dt$$

$$= \int_0^1 5(2t)\,2\,dt = 10\,t^2\Big|_0^1 = 10$$

C_2: $x = 2\cos(t),\ y = 2\sin(t), \quad 0 \le t \le \pi/2$

$x' = -2\sin(t),\ y' = 2\cos(t)$

$ds = \sqrt{4\sin^2(t) + 4\cos^2(t)}\,dt = 2\,dt$

NOTE: ORIENTATION DOESN'T MATTER!

BUT YOU DO NEED A PROPER FORWARD IN TIME PARAMETERIZATION OF C_2.

$$\int_{C_2} T(x,y)\,ds = \int_0^{\pi/2} \left(\tfrac{1}{3}4\cos^2(t)\,2\sin(t) + 5\sqrt{4}\right) 2\,dt$$

$$= \int_0^{\pi/2} \tfrac{16}{3}\cos^2(t)\sin(t)\,dt + \int_0^{\pi/2} 20\,dt$$

$u = \cos(t)$

$du = -\sin(t)\,dt$

$$= \tfrac{16}{3}\int_0^1 u^2\,du + 10\pi$$

$$= \tfrac{16}{9}u^3\Big|_0^1 + 10\pi = \tfrac{16}{9} + 10\pi$$

$$\text{AVERAGE} = \boxed{\frac{10 + \frac{16}{9} + 10\pi}{2+\pi} = \frac{106 + 90\pi}{18 + 9\pi} \approx 8.40084\ {}^\circ C}$$