16.3: The Fundamental Theorem for Line Integrals over Vector Fields

Given a vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{i}$ on $\mathbb{R}^2$, we say $\mathbf{F}$ is **conservative** if

$$\mathbf{F} = \nabla f, \text{ for some surface } z = f(x, y).$$

We call $f(x, y)$ a *potential* function for $\mathbf{F}$. **NOT all vector fields are conservative.** But here is what we found, if the vector field is conservative.

1. Under fairly general conditions (the domain has to be open and simply-connected and the first partials must be continuous), $\mathbf{F}$ is conservative if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

   This gives a quick **test** if a field is conservative. (The conditions for $\mathbb{R}^3$ will be given in section 16.5).

2. If is conservative, then the big theorem was:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1),$$

   where (x_1, y_1) and (x_2, y_2) are the start and end points, respectively, for the curve C. In other words, the work done by the force field in equal to the change in the potential function from the start point to the end point.

3. Some immediate consequences were:

   - $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths that start at point A and end at point B. This is called *path independence.*
   - $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any *closed* curve C.
   - In fact, we found that these conditions are only true in general for conservative vector fields (on open connected regions). So these properties characterize conservative vector fields.

Finding $f(x,y)$ such that $\nabla f = \mathbf{F}$:
Provided our vector field has passed the test and we know it is conservative, then we can start looking for a potential function f. The goal is to find $f(x,y)$ such that

$$f_x(x,y) = P(x,y) \text{ and } f_y(x,y) = Q(x,y).$$

So we need to integrate $P(x,y)$ with respect to x and $Q(x,y)$ with respect to y to find $f(x,y)$. Here is a systematic way to summarize this process:

1. Integrate $f_x(x,y) = P(x,y)$ with respect to get $f(x,y) = \int P(x,y)dx + g(y)$ (where $g(y)$ is some function involving only y, that is yet to be found).
2. Compute $f_y(x,y)$ for what we have so far (which will include $g'(y)$).
3. Since $f_y(x,y) = Q(x,y)$, we simplify to see what $g'(y)$ should be, then integrate to get $g(y)$.
4. We're DONE: $f(x,y) = \displaystyle\int P(x,y)dx + g(y)$.

Here's some examples:

- $\mathbf{F} = \langle 3x^2y, x^3+y \rangle$. This passes the test ($\frac{\partial P}{\partial y} = 3x^2 = \frac{\partial Q}{\partial x}$), it is a conservative vector field. So we start integrating:
  1. $f(x,y) = \displaystyle\int 3x^2y\ dx = x^3y + g(y)$.
  2. Since $f_y(x,y) = x^3 + y$, we want $x^3 + g'(y) = x^3 + y$. So $g'(y) = y$.
  3. Hence, $g(y) = \displaystyle\int y\ dy = \frac{1}{2}y^2 + k$ for some constant k.
  4. $f(x,y) = x^3y + \frac{1}{2}y^2 + k$.
- $\mathbf{F} = \langle \cos(y) + \sin(x), 3y^2 - x\sin(y) \rangle$. This passes the test ($\frac{\partial P}{\partial y} = -\sin(y) = \frac{\partial Q}{\partial x}$), it is a conservative vector field. So we start integrating:
  1. $f(x,y) = \displaystyle\int \cos(y) + \sin(x)\ dx = x\cos(y) - \cos(x) + g(y)$.
  2. Since $f_y(x,y) = 3y^2 - x\sin(y)$, we want $-x\sin(y) + g'(y) = 3y^2 - x\sin(y)$. So $g'(y) = 3y^2$.
  3. Hence, $g(y) = \displaystyle\int 3y^2\ dy = y^3 + k$ for some constant k.
  4. $f(x,y) = x\cos(y) - \cos(x) + y^3 + k$.

If it was a problem over $\mathbb{R}^3$ and we were told that the vector field is conservative, then the process is similar:

1. Integrate $f_x(x,y,z) = P(x,y,z)$ with respect to get $f(x,y,z) = \int P(x,y,z)dx + g(y,z)$ (where $g(y,z)$ is some function involving only y and z, that is yet to be found).
2. Compute $f_y(x,y,z)$ for what we have so far (which will include $g_y(y,z)$).
3. Since $f_y(x,y,z) = Q(x,y,z)$, we simplify to see what $g_y(y,z)$ should be and integrate with respect to y to get $g(y,z) =$ 'an integral' $+ h(z)$.(where $h(z)$ is some function involving only z, that is yet to be found).
4. Compute $f_z(x,y,z)$ for what we have so far (which will include $h'(z)$).
5. Since we want $f_z(x,y,z) = R(x,y,z)$, we simplify to see what $h'(z)$ should be, then integrate with respect to z to get $h(z)$.
6. We're DONE: $f(x,y,z) = \displaystyle\int P(x,y)dx + g(y,z) + h(z)$.

Here's an example:

- $\mathbf{F} = \langle e^{-2y}, z^3 - 2xe^{-2y}, 3(y+1)z^2 \rangle$. I am telling you this is conservative (in 16.5 we will discuss the test). So we start integrating:
  1. $f(x,y,z) = \displaystyle\int e^{-2y}\ dx = xe^{-2y} + g(y,z)$.
  2. Since $f_y(x,y,z) = z^3 - 2xe^{-2y}$, we want $-2xe^{-2y} + g_y(y,z) = z^3 - 2xe^{-2y}$. So $g_y(y,z) = z^3$.
  3. Hence, $g(y,z) = \displaystyle\int z^3\ dy = yz^3 + h(z)$. At this point, we know $f(x,y,z) = xe^{-2y} + yz^3 + h(z)$.
  4. Since $f_z(x,y,z) = 3(y+1)z^2$, we want $0 + 3yz^2 + h'(z) = 3(y+1)z^2$. So $h'(z) = 3z^2$.
  5. Hence, $h(z) = \displaystyle\int 3z^2\ dy = z^3 + k$ for some constant k.
  6. $f(x,y,z) = xe^{-2y} + yz^3 + z^3 + k$.
- $\mathbf{F} = \left\langle \frac{1}{x-y}, \frac{1}{y-x}, 2z \right\rangle$. I am telling you this is conservative. So we start integrating:
  1. $f(x,y,z) = \displaystyle\int \frac{1}{x-y}\ dx = \ln|x-y| + g(y,z)$.
  2. Since $f_y(x,y,z) = \frac{1}{y-x}$, we want $-\frac{1}{x-y} + g_y(y,z) = \frac{1}{y-x}$. So $g_y(y,z) = 0$.
  3. Hence, $g(y,z) = \displaystyle\int 0\ dy = h(z)$ (a constant plus a function of z). At this point, we know $f(x,y,z) = \ln|x-y| + h(z)$.
  4. Since $f_z(x,y,z) = 2z$, we want $0 + h'(z) = 2z$. So $h'(z) = 2z$.
  5. Hence, $h(z) = \displaystyle\int 2z\ dy = z^2 + k$ for some constant k.
  6. $f(x,y,z) = \ln|x-y| + z^2 + k$.