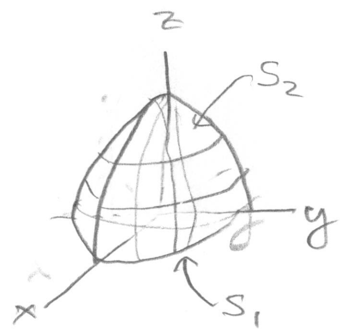
SURFACE INTEGRALS FOR SCALAR FIELDS

4 SET-UP EXAMPLES

I $\boxed{\iint_S f(x,y,z)\,dS} = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$

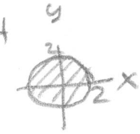
$S: z = g(x,y) \longrightarrow \;\; = \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$

Ex Let S be the boundary of the solid between $z = 4 - x^2 - y^2$ and the xy-plane.



ANS

- S_1 = the bottom $\quad z = 0 \Rightarrow \quad x^2+y^2=4$



PARAMETERIZE:

$x = v\cos(u) \qquad 0 \le u \le 2\pi$

$y = v\sin(u) \qquad 0 \le v \le 2$

$z = 0$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin(u) & v\cos(u) & 0 \\ \cos(u) & \sin(u) & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + (-v\sin^2(u) - v\cos^2(u))\,k$

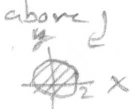
$= \langle 0, 0, -v\rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{0^2 + 0^2 + (-v)^2} = v$

$\iint_{S_1} f(x,y,z)\,dS = \int_0^2 \int_0^{2\pi} f(v\cos(u), v\sin(u), 0)\; v\,du\,dv$

ALSO YOU CAN USE
$x = x$
$y = y$
$z = 0$

- S_2 = the top $\quad z = 4 - x^2 - y^2 \qquad 0 \le z \le 4 \leftarrow$ or just note above



PARAMETERIZE

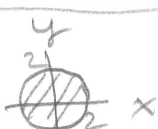
OPTION 1: $x = x,\ y = y,\ z = 4 - x^2 - y^2$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + (-2x)^2 + (-2y)^2}$

$= \sqrt{1 + 4x^2 + 4y^2}$

$\iint_{S_2} f(x,y,z)\,dS$

$= \iint_D f(x,y,4-x^2-y^2)\sqrt{1+4x^2+4y^2}\,dA$



USE POLAR

OPTION 2: $x = v\cos(u),\ y = v\sin(u),\ z = 4 - v^2$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin(u) & v\cos(u) & 0 \\ \cos(u) & \sin(u) & -2v \end{vmatrix} \qquad \begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le 2 \end{matrix}$

$= -2v^2\cos(u)\,\vec{i} - 2v^2\sin(u)\,\vec{j} - v\,\vec{k}$

$= \langle -2v^2\cos(u), -2v^2\sin(u), -v\rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{4v^4\cos^2(u) + 4v^4\sin^2(u) + v^2}$

$= \sqrt{4v^4 + v^2} = v\sqrt{4v^2+1}$

$\iint_{S_2} f(x,y,z)\,dS$

$= \int_0^2 \int_0^{2\pi} f(v\cos(u), v\sin(u), 4-v^2)\; v\sqrt{4v^2+1}\,dA$

$\iint_S f(x,y,z)\,dS = \iint_{S_1} f(x,y,z)\,dS + \iint_{S_2} f(x,y,z)\,dS$

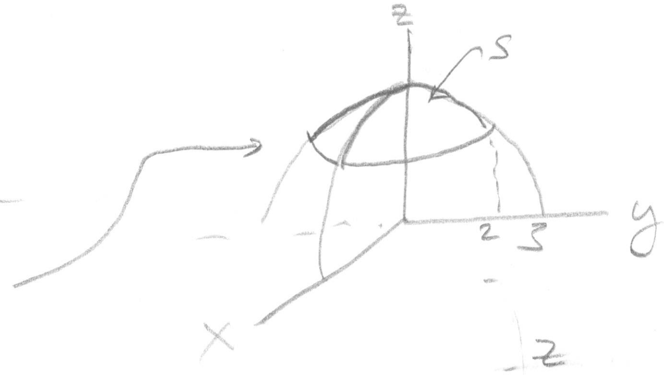
Ex

Let S be the upper hemisphere, centred at the origin, of radius 3, and inside the cylinder $x^2+y^2=4$.

ANS

The sphere $x^2+y^2+z^2=9$
and $-\quad x^2+y^2 \quad = 4$
intersect when $z^2=5$

$z=\sqrt{5}$

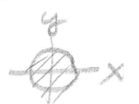


PARAMETERIZE

OPTION 1: $x=x,\ y=y$

$z=\sqrt{9-x^2-y^2}$

ABOVE $x^2+y^2\leq 1$



$$|\vec{r}_x\times\vec{r}_y|=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

$$=\sqrt{1+\left(\frac{-2x}{2\sqrt{9-x^2-y^2}}\right)^2+\left(\frac{-2y}{2\sqrt{9-x^2-y^2}}\right)^2}$$

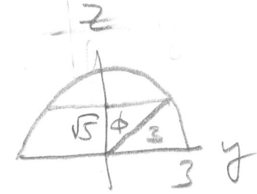
$$=\sqrt{1+\frac{x^2+y^2}{\sqrt{9-x^2-y^2}}}$$

$$\iint_S f(x,y,z)\,dS$$

$$=\iint_D f\left(x,y,\sqrt{9-x^2-y^2}\right)\sqrt{1+\frac{x^2+y^2}{\sqrt{9-x^2-y^2}}}\,dA$$

$$=\int_0^{2\pi}\int_0^1 f\left(r\cos(\theta),r\sin(\theta),\sqrt{9-r^2}\right)\sqrt{1+\frac{r^2}{\sqrt{9-r^2}}}\,r\,dr\,d\theta$$

OPTION 2:



$x=3\sin\phi\cos\theta$

$y=3\sin\phi\sin\theta$

$z=3\cos\phi$

$0\leq\theta\leq 2\pi$

$0\leq\phi\leq\cos^{-1}(\sqrt{5}/3)$

$\cos\phi=\frac{\sqrt{5}}{3}$

$\phi=\cos^{-1}(\sqrt{5}/3)$

$|\vec{r}_\phi\times\vec{r}_\theta|=3^2\sin\phi$

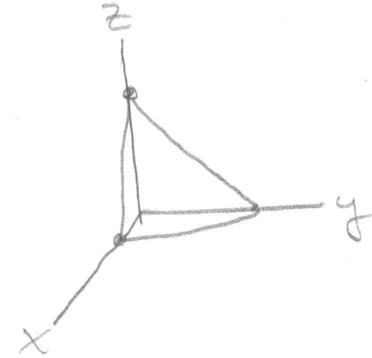
$$\iint_S f(x,y,z)\,dS$$

$$=\int_0^{2\pi}\int_0^{\cos^{-1}(\sqrt{5}/3)} f(3\sin\phi\cos\theta,3\sin\phi\sin\theta,3\cos\phi)\,9\sin\phi\,d\phi\,d\theta$$

Ex)

Let S be the triangle with corners

$P(1,0,0)$, $Q(0,5,0)$, $R(0,0,10)$



ANS

EQUATION FOR THE PLANE

$\overrightarrow{PQ} = \langle -1, 5, 0 \rangle$ $\quad \overrightarrow{PR} = \langle -1, 0, 10 \rangle$

$$\text{normal} = \vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 5 & 0 \\ -1 & 0 & 10 \end{vmatrix}$$

$$= \langle 50, 10, 5 \rangle$$

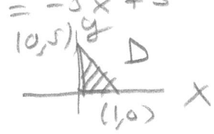
$50(x-1) + 10y + 5z = 0$ divide by 5 to simplify

$10(x-1) + 2y + z = 0$

$$z = -10x - 2y + 10$$

PROJECTION ONTO xy-plane $(z=0) \Rightarrow$ $2y = -10x + 10$

$y = -5x + 5$



PARAMETERIZE: $x = x$, $y = y$, $z = -10x - 2y + 10$

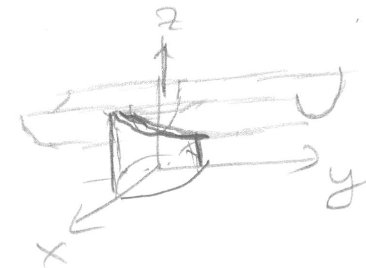
$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + (-10)^2 + (-2)^2}$$

$$= \sqrt{1 + 100 + 4} = \sqrt{105}$$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,-10x-2y+10)\sqrt{105}\,dA$$

$$= \int_0^1 \int_0^{-5x+5} f(x,y,-10x-2y+10)\sqrt{105}\,dy\,dx$$

Ex Let S be the part of the cylinder $x^2+y^2=4$ that is above $z=0$ and below $z=x^2+2$.



ANS

PARAMETERIZE THE CYLINDER

$x = 2\cos(u)$ $\quad 0 \le u \le 2\pi$

$y = 2\sin(u)$ $\quad 0 \le v \le x^2+2 = 4\cos^2(u)+2$

$z = v$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin(u) & 2\cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos(u)\vec{i} + 2\sin(u)\vec{j} + 0\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4\cos^2(u)+4\sin^2(u)+0} = 2$$

$$\iint_S f(x,y,z)\,dS = \int_0^{2\pi}\int_0^{4\cos^2(u)+2} f(2\cos(u), 2\sin(u), v)\, 2\, dv\, du$$

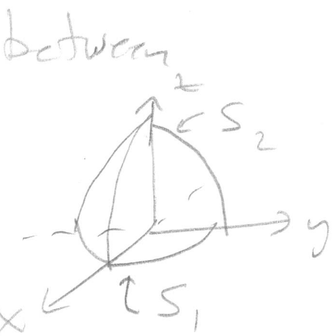
SURFACE INTEGRALS OF VECTOR FIELDS

$\boxed{\text{II}}$ $$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA}$$

$$\boxed{\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle}$$

SAME 4 SET-UP SURFACES

Ex) Let S be the boundary of the solid between $z = 4 - x^2 - y^2$ and the xy-plane



ANS) Compute the total outward flux

- S_1 = the bottom

$x = v\cos(u)$ $0 \le u \le 2\pi$
$y = v\sin(u)$ $0 \le v \le 2$
$z = 0$

$\vec{r}_u \times \vec{r}_v = \langle 0, 0, -v \rangle$ ↑ downward orientation which is outward ✓

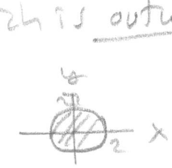
$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \int_0^2 \int_0^{2\pi} \vec{F}(v\cos(u), v\sin(u), 0) \cdot \langle 0, 0, -v \rangle\, du\, dv$$

- S_2 = the top

$x = x,\ y = y,\ z = 4 - x^2 - y^2$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle$$

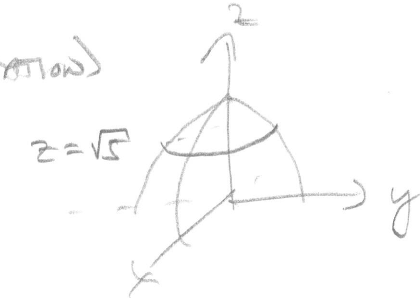
upward orientation which is outward ✓



$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(x, y, 4 - x^2 - y^2) \cdot \langle 2x, 2y, 1 \rangle\, dA$$

USE POLAR NEXT

Ex) Let S be the upper hemisphere, centered at the origin, of radius 3 and inside the cylinder $x^2+y^2=4$. (UPWARD ORIENTATION)



ANS

WITH SPHERICAL COORDINATES PARAMETERIZATION

$$x = 3\sin\phi\cos\theta,\ y = 3\sin\phi\sin\theta,\ z = 3\cos\phi$$

$$0 \le \theta \le 2\pi,\quad 0 \le \phi \le \cos^{-1}(\sqrt{5}/3)$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3\cos\phi\cos\theta & 3\cos\phi\sin\theta & -3\sin\phi \\ -3\sin\phi\sin\theta & 3\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \langle 9\sin^2\phi\cos\theta,\ 9\sin^2\phi\sin\theta,\ 9\sin\phi\cos\phi \rangle$$

OUTWARD /UPWARD ORIENTATION

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\cos^{-1}(\sqrt{3}/3)} \vec{F}(3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi)\cdot\langle 9\sin^2\phi\cos\theta, 9\sin^2\phi\sin\theta, 9\sin\phi\cos\phi\rangle\, d\phi\, d\theta$$

Ex) Let S be the triangle with corners $P(1,0,0)$, $Q(0,5,0)$, $R(0,0,10)$. UPWARD ORIENTATION

ANS

EQUATION FOR THE PLANE

$$z = -10x - 2y + 10$$

$$x = x,\ y = y,\ z = -10x - 2y + 10$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -10 \\ 0 & 1 & -2 \end{vmatrix} = \langle 10, 2, 1 \rangle$$

UPWARD ORIENTATION

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(x,y,-10x-2y+10)\cdot\langle 10,2,1\rangle\, dA$$

$$= \int_0^1\int_0^{-5x+5} \vec{F}(x,y,-10x-2y+10)\cdot\langle 10,2,1\rangle\, dy\, dx$$

Ex) Let S be the part of the cylinder $x^2+y^2=4$ that is above $z=0$ and below $z=x^2+2$. (OUTWARD ORIENTATION)

ANS

$x = 2\cos(u)$ $\qquad 0 \le u \le 2\pi$

$y = 2\sin(u)$ $\qquad 0 \le v \le x^2+2 = 4\cos^2(u)+2$

$z = v$

$$\underbrace{\vec{r}_u \times \vec{r}_v}_{\text{position vector for circle, points outward}} = \langle 2\cos(u), 2\sin(u), 0 \rangle$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(2\cos(u), 2\sin(u), v) \cdot \langle 2\cos(u), 2\sin(u), 0 \rangle \, dA$$

$$= \int_0^{2\pi} \int_0^{4\cos^2(u)+2} \vec{F}(2\cos(u), 2\sin(u), v) \cdot \langle 2\cos(u), 2\sin(u), 0 \rangle \, dv\, du$$