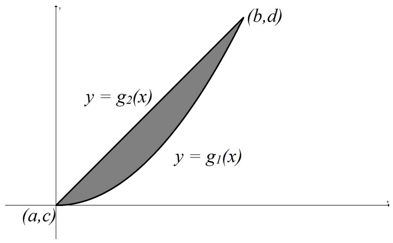
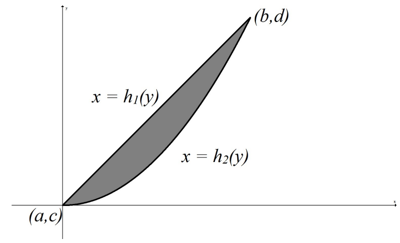
Discussion of the Proof of Green's Theorem (from 16.4)

Green's Theorem states: On a positively oriented, simple closed curve C that encloses the region D

$$\int_C P\,dx + Q\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

The general proof goes beyond the scope of this course, but in a simple situation we can prove it. Consider the region:





We will show that $\int_C P\,dx = -\iint\limits_D \frac{\partial P}{\partial y} dA$ and $\int_C Q\,dy = \iint\limits_D \frac{\partial Q}{\partial y} dA$

The double integral looks like:

$$\iint\limits_D \frac{\partial P}{\partial y} dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} dy dx = \int_a^b P(x, g_2(x)) - P(x, g_1(x)) dx$$

Now, let C_1 be the lower part of the closed curve and C_2 the upper part. We can parameterize the line integrals by: C_1: $x = t$, $y = g_1(t)$, $a \le t \le b$ and $-C_2$: $x = t$, $y = g_2(t)$, $a \le t \le b$.
Notice that the second gives the opposite orientation which is why I noted that this was $-C_2$. Using these parameterization we get

$$\int_C P(x,y)dx = \int_{C_1} P(x,y)dx + \int_{C_2} P(x,y)dx = \int_a^b P(t, g_1(t))dt - \int_a^b P(t, g_2(t))dt.$$

This is identical, with opposite sign, to the double integral result and we have

$$\int_C P dx = -\iint\limits_D \frac{\partial P}{\partial y} dA.$$

And similarly for $Q(x,y)$:

$$\iint\limits_D \frac{\partial Q}{\partial x} dA = \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x} dx dy = \int_c^d Q(h_2(y), y) - Q(h_1(y), y) dy$$

Now, let C_1 be the right part of the closed curve and C_2 the left part (this is the same as above, I'm just referencing the perspective shift). We can parameterize the line integrals by:
C_1: $x = h_2(t)$, $y = t$, $c \le t \le d$ and $-C_2$: $x = h_1(t)$, $y = t$, $c \le t \le d$.
Again, notice that the second gives the opposite orientation which is why I noted that this was $-C_2$. Using these parameterization we get

$$\int_C Q(x,y)dy = \int_{C_1} Q(x,y)dy + \int_{C_2} Q(x,y)dy = \int_c^d Q(h_2(t), t)dt - \int_c^d Q(h_1(t), t)dt.$$

This is identical, with the same sign, to the double integral result and we have

$$\int_C Q dy = \iint\limits_D \frac{\partial Q}{\partial x} dA.$$